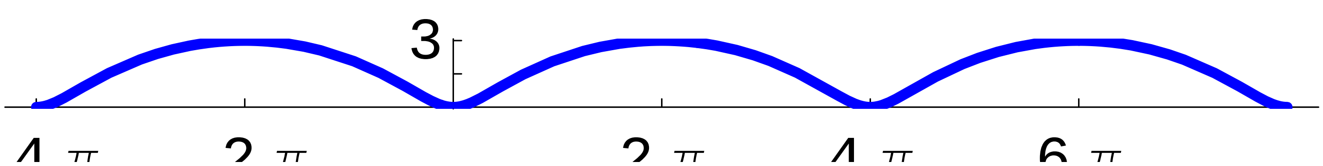
"Korak po korak" rješenje 3. kontrolne zadaća iz Matematike II

Zadatak 1

Riješite sljedeći binomni integral: $\int \frac{\sqrt[3]{1+\sqrt[4]{x}}}{\sqrt{x}}dx.$

$$\begin{aligned}
\int \frac{\sqrt[3]{1+\sqrt[4]{x}}}{\sqrt{x}}dx &= \left| \begin{array}{lll} m=-\frac{1}{2}; & \frac{m+1}{n}=2; & dx = d\left((z^3-1)^4\right) = \\ n=\frac{1}{4}; & 1+x^{\frac{1}{4}}=z^3; & =4(z^3-1)^3\cdot 3z^2dz = \\ p=\frac{1}{3}; & x=(z^3-1)^4; & =12z^2(x^3-1)^3dz \end{array} \right| = \\
&= \int \left((z^3-1)^4\right)^{-\frac{1}{2}} \cdot \left(z^3\right)^{\frac{1}{3}} \cdot 12z^2(z^3-1)^3dz \\
&= 12\int (z^3-1)^{-2}\cdot z\cdot z^2(z^3-1)^3dz = \\
&= 12\int (z^3-1)z^3dz = 12\int (z^6-z^3)dz = 12\left(\frac{z^7}{7}-\frac{z^4}{4}\right) = \\
&= 12\left(\frac{\left(\sqrt[3]{1+\sqrt[4]{x}}\right)^7}{7} - \frac{\left(\sqrt[3]{1+\sqrt[4]{x}}\right)^4}{4}\right) + C
\end{aligned}$$

Zadatak 2 *Odredite površinu omeđenu jednom granom trohoide* $\begin{array}{lll} x &=& 2t-\sin t \\ y &=& 2-\cos t \end{array}, t\in[0,2\pi].$

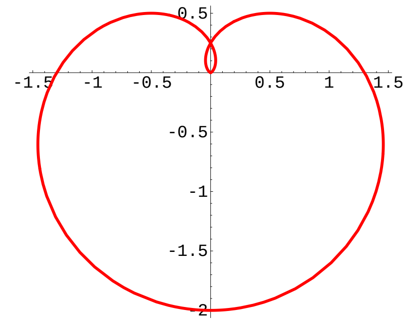


$$\begin{aligned}
P &= \int_0^{2\pi} y(t)x'(t)dt = \int_0^{2\pi}(2-\cos t)(2-\cos t)dt = \int_0^{2\pi}(2-\cos t)^2dt = F(2\pi)-F(0) = \\
&= 4\cdot 2\pi - 4\sin(2\pi) + \frac{sin^2(2\cdot 2\pi)}{4} + \frac{2\pi}{2} - \left(4\cdot 0 - 4\sin 0 + \frac{\sin^2(2\cdot 0)}{4} + \frac{0}{2}\right) = 9\pi
\end{aligned}$$

$$\begin{aligned}
F(t) &= \int (2-\cos t)^2dt = \int (4-4\cos t+\cos^2 t)dt = 4t-4\sin t + \underbrace{\int \cos^2 t\,dt}_{I_1} = \\
&= 4t - 4\sin t + \frac{sin(2t)}{4} + \frac{t}{2}
\end{aligned}$$

$$\begin{aligned}
I_1 &= \int \cos^2 t\,dt = \left| \left. \begin{array}{l} \cos(2t) = \cos^2 t - \sin^2 t \\ cos^2 t = \cos(2t) + \sin^2 t = \\ = \cos(2t) + 1 - \cos^2 t \end{array} \right\} \Rightarrow \cos^2 = \frac{\cos(2t)}{2} + \frac{1}{2} \right| = \\
&= \int \left(\frac{\cos(2t)}{2} + \frac{1}{2}\right)dt = \int \frac{\cos(2t)}{2}\,d(2t)\cdot\frac{1}{2} + \int \frac{1}{2}dt = \frac{\sin(2t)}{4} + \frac{t}{2}
\end{aligned}$$

Zadatak 3 *Odredite duljinu čitave krivulje $r = 2\sin^3\frac{\varphi}{3}$ koja je skup točaka (r, φ) kada se φ mijenja od 0 do 3π.*

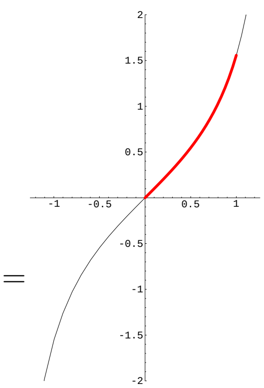


$r' = 2\sin^2\frac{\varphi}{3}\cos\frac{\varphi}{3}$

$$\begin{aligned} s &= \int_0^{3\pi}\sqrt{[r(\varphi)]^2+[r'(\varphi)]^2}d\varphi = \int_0^{3\pi}\sqrt{4\sin^6\frac{\varphi}{3}+4\sin^4\frac{\varphi}{3}\cos^2\frac{\varphi}{3}}\,d\varphi = \int_0^{3\pi}\sqrt{4\sin^4\frac{\varphi}{3}\left(\sin^2\frac{\varphi}{3}+\cos^2\frac{\varphi}{3}\right)}d\varphi = \\ &= 2\int_0^{3\pi}\sin^2\frac{\varphi}{3}d\varphi = \left| \begin{array}{l} \frac{\varphi}{3}=t \\ d\varphi = 3dt \\ t_1=0;\ t_2=\pi \end{array} \right| = 6\int_0^{\pi}\sin^2 t\,dt = 6\left(F(\pi)-F(0)\right) = \\ &= 6\left(\frac{\pi}{2}-\frac{\sin(2\cdot\pi)}{4}-\left(\frac{0}{2}-\frac{\sin(2\cdot 0)}{4}\right)\right) = 3\pi \text{ mj. jed.} \end{aligned}$$

$$\begin{aligned} F(t) &= \int\sin^2 t\,dt = \left| \left.\begin{array}{l} \cos(2t)=\cos^2 t-\sin^2 t; \\ \sin^2 t = \cos^2 t - \cos(2t) = \\ = 1-\sin^2 t-\cos(2t) \end{array}\right\} \Rightarrow\ \sin^2 t = \tfrac{1}{2}-\tfrac{\cos(2t)}{2} \right| = \int\left(\frac{1}{2}-\frac{\cos(2t)}{2}\right)dt \\ &= \frac{t}{2}-\frac{1}{2}\int\cos(2t)d(2t)\cdot\frac{1}{2} = \frac{t}{2}-\frac{\sin(2t)}{4} \end{aligned}$$

Zadatak 4 *Napišite formule kojima se mogu računati volumeni rotacijskih tijela (u ovisnosti o načinu zadavanja krivulje). Odredite volumen tijela koje nastaje rotacijom oko osi apscisa (osi x) dijela tangentoide $y = \operatorname{tg} x$ od $x = 0$ do $x = 1$. Nacrtajte sliku!*



$$\begin{aligned} V &= \pi\int_0^1 \operatorname{tg}^2 x dx = \pi\int_0^1\frac{\sin^2 x}{\cos^2 x}dx = \pi\int_0^1\frac{1-\cos^2 x}{\cos^2 x}dx = \pi\int_0^1\left(\frac{1}{\cos^2 x}-1\right)dx = \\ &= \pi\left(\operatorname{tg} x - x\right)_0^1 = \pi(\operatorname{tg} 1 - 1 - (\operatorname{tg} 0 - 0) = \pi(\operatorname{tg} 1 - 1) \end{aligned}$$

Zadatak 5 *Po kojim formulamo se mogu računati oplošja rotacijskih tijela? Odredite površinu rotacijske plohe koja nastaje rotacijom oko osi x parabole $y = \sqrt{x}$ od $x = 2$ do $x = 3$. Nacrtajte sliku!!*

$$\begin{aligned} P &= 2\pi\int_2^3\sqrt{x}\cdot\sqrt{1+\left(\left(\sqrt{x}\right)'\right)^2}dx = 2\pi\int_2^3\sqrt{x\cdot\left(1+\left(\frac{1}{2\sqrt{x}}\right)^2\right)}dx = \\ &= 2\pi\int_2^3\sqrt{x\cdot\left(1+\frac{1}{4x}\right)}dx = 2\pi\int_2^3\sqrt{\cancel{x}\cdot\frac{4x+1}{4\cancel{x}}}\,dx = \pi\int_2^3\sqrt{4x+1}\,dx \\ &= \pi\int_2^3(4x+1)^{\frac{1}{2}}d(4x+1)\cdot\frac{1}{4} = \frac{\pi}{4}\frac{(4x+1)^{\frac{3}{2}}}{\frac{3}{2}}\Bigg|_2^3 = \frac{\pi}{6}\sqrt{(4x+1)^3}\Bigg|_2^3 = \\ &= \frac{\pi}{6}\left(\sqrt{(4\cdot 3+1)^3}-\sqrt{(4\cdot 2+1)^3}\right) = \frac{\pi}{6}\left(13\sqrt{13}-27\right) \end{aligned}$$