RELATIVE RESIDUAL BOUNDS FOR INDEFINITE SINGULAR HERMITIAN MATRICES *

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Abstract. This paper presents a linear and quadratic residual bound for indefinite possible singular Hermitian matrix. These theorems are proper generalization of results on a semi-definite Hermitian matrix *SIAM Journal on Matrix Analysis and Appl.*, 18:1:21–29 (1997). The bounds here contains an extra factor which depends on norm of J-unitary matrix, where J is diagonal matrix with ± 1 on its diagonal.

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1. Introduction. Let $H \in \mathbf{C}^{n\times n}$ be a Hermitian matrix, $X \in \mathbf{C}^{n\times m}$ be an orthonormal matrix, and

$$M = X^*HX, \qquad R = HX - XM, \qquad \mathcal{X} = \mathcal{R}(X). \tag{1.1}$$

Furthermore, let

$$\lambda_1 \geq \ldots \geq \lambda_n \qquad \text{and} \qquad \mu_1 \geq \ldots \geq \mu_m, \tag{1.2}$$

be the eigenvalues of H and M, respectively.

In this paper we present a linear and quadratic residual bound for indefinite possible singular Hermitian matrix.

In [8] has been presented the following linear residual bound for non-singular indefinite Hermitian matrices:

THEOREM 1.1. *Let $H = LJL^*$, where L and J are non-singular and J is diagonal with ± 1 on its diagonal. Let*

$$\mathcal{Y}_L = JL^*\mathcal{X}, \qquad \mathcal{Z}_L = L^{-1}\mathcal{X},$$

and let ψ be the maximal acute principal angle between $\mathcal{Y}_L$ and $\mathcal{Z}_L$. There are at least m eigenvalues λ_{i_k}, $k = 1, \ldots, m$ of H for which

$$\frac{|\lambda_{i_k} - \mu_k|}{|\lambda_{i_k}|} \leq \kappa(V)\frac{\sin\psi}{1-\sin\psi} \qquad k = 1, \ldots, m, \tag{1.3}$$

provided that right hand side in (1.3) is less than one. Here V is J-unitary matrix which diagonalizes the pair (L^*L, J), that is, $V^*L^*LV = |\Lambda|$ and $V^*JV = J$.

We are going to bound $|\lambda_{i_k} - \mu_k|/|\mu_{i_k}|$ provided that the matrix H can be semidefinite. Our bounds are proper generalization of linear residual bounds for positive semidefinite matrices presented in [1].

On the other hand, all exited quadratic residual bounds for general Hermitian matrices belong to classical perturbation theory.

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Let $\sigma(H)$ denote the spectra of H. The first result is due to Sun [5].

THEOREM 1.2 (Sun). *Let $\mathcal{Y} = \mathcal{R}(Y)$ be an invariant subspace of H with orthonormal basis $Y \in \mathbf{C}^{n\times m}$. Let $\lambda_{j_1} \geq \ldots \geq \lambda_{j_m}$ be the eigenvalues of Y^*HY, and $\Lambda_{\mathcal{Y}} = \mathrm{diag}(\lambda_{j_1}, \ldots, \lambda_{j_m})$, $\Lambda_{\mathcal{X}} = \mathrm{diag}(\mu_1, \ldots, \mu_m)$. If for some $\alpha, \beta \in \mathbf{R}$ and $\delta_0 > 0$, $\sigma(M) \subset [\alpha, \beta]$, $\sigma(H) \setminus \sigma(Y^*HY) \subset (-\infty, \alpha - \delta_0] \cup [\beta + \delta_0, +\infty)$, (or vice versa) and if $\rho \equiv \|R\|_2/\delta_0 < 1$, then for any unitary invariant norm $\|\cdot\|$,*

$$\|\Lambda_{\mathcal{Y}} - \Lambda_{\mathcal{X}}\| \leq \frac{1}{\sqrt{1-\rho^2}} \cdot \frac{\|R\|_2\|R\|}{\delta_0}.$$

The second result is due to Mathias [4], and this result is generalization of result obtained by Theorem 1.2.

Let

$$H = \begin{bmatrix} A & R_0 \\ R_0^* & B \end{bmatrix} \quad \text{and} \quad \widetilde{H} = \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}$$

be Hermitian matrices. For measure of separation between eigenvalues λ_k, $k = 1, \ldots, n$ of the matrix H from eigenvalues $\mu_i(B)$ of the matrix B we define

$$\delta_k \equiv \min_{i=1,\ldots,n} |\lambda_k - \mu_i(B)|.$$

For the measure of separation between eigenvalues $\widetilde{\lambda}_k$, $k = 1, \ldots, n$ of the matrix $\widetilde{H}$ from eigenvalues $\mu_i(B)$ of the matrix B we use

$$\widetilde{\delta}_k \equiv \min_{i=1,\ldots,n} |\widetilde{\lambda}_k - \mu_i(B)|.$$

THEOREM 1.3. *[4, Theorem 1] If $\lambda_k \notin \sigma(B)$, then*

$$|\lambda_k - \widetilde{\lambda}_k| \leq \delta_k^{-1}\|R_0\|^2,$$

while if $\widetilde{\lambda}_k \notin \sigma(B)$, then

$$|\lambda_k - \widetilde{\lambda}_k| \leq \widetilde{\delta}_k^{-1}\|R_0\|^2.$$

Similarly as in the linear case our quadratic bound is a proper generalization of the quadratic residual bound for positive semidefinite matrices presented in [1].

2. Linear residual bound. In this section we present the relative residual bounds for indefinite, possible singular, Hermitian matrices. First we will separate null subspace of the matrix H from the rest of the subspaces.

Let $H = GJG^*$ be an indefinite Hermitian matrix, and let $\mathcal{X}$ be $m-$dimensional subspace of $\mathbf{C}^n$. Let $X = \begin{bmatrix} X_1 & X_2 \end{bmatrix}$ and $X_\perp = \begin{bmatrix} X_{\perp,1} & X_{\perp,2} \end{bmatrix}$ be orthonormal bases of $\mathcal{X}$, $\mathcal{X}_\perp$, respectively, such that $G^*X_1 = 0$ and $G^*X_{\perp,2} = 0$ and

$$M = \begin{bmatrix} X_1 & X_2 \end{bmatrix}^* H \begin{bmatrix} X_1 & X_2 \end{bmatrix} = \begin{array}{c} \begin{bmatrix} 0 & \\ & \Lambda_1 \end{bmatrix} \end{array} \begin{array}{l} m - r_M \\ r_M \end{array}, \tag{2.1}$$

$$N = \begin{bmatrix} X_{\perp,1} & X_{\perp,2} \end{bmatrix}^* H \begin{bmatrix} X_{\perp,1} & X_{\perp,2} \end{bmatrix} = \begin{array}{c} \begin{bmatrix} \Lambda_2 & \\ & 0 \end{bmatrix} \end{array} \begin{array}{l} r_N \\ n - m - r_M \end{array}, \tag{2.2}$$

where $r_M = \text{rank}\,(M)$, $r_N = \text{rank}\,(N)$.

Now we can write

$$\begin{bmatrix} X & X_\perp \end{bmatrix}^* H \begin{bmatrix} X & X_\perp \end{bmatrix} = \begin{array}{c} \begin{bmatrix} 0 & & \\ & \widehat{H} & \\ & & 0 \end{bmatrix} \end{array} \begin{array}{l} m - r_M \\ r_M + r_N \\ n - m - r_N \end{array} ,$$

where

$$\widehat{H} = \begin{bmatrix} \Lambda_1 & K^* \\ K & \Lambda_2 \end{bmatrix} \begin{array}{l} r_M \\ r_N \end{array} . \tag{2.3}$$

The following theorem contains relative perturbation bound for the eigenvalues of H and Rayleigh-Ritz approximations of eigenvalues of H, that is, eigenvalues of M, where M is given by (2.1) .

THEOREM 2.1. *Let $H = GJG^*$, be indefinite Hermitian matrix (possibly singular), where J is diagonal matrix with ± 1 on its diagonal. Let X and $X_\perp$ be orthonormal matrices as in (2.1) and (2.2), if we define $K_S = |\Lambda_2|^{-\frac{1}{2}} K |\Lambda_1|^{-\frac{1}{2}}$ then*

$$\frac{|\lambda_{i_k} - \mu_{m-r_M+k}|}{|\mu_{m-r_M+k}|} \leq \|K_S\|, \qquad k = 1, \ldots, r_M, \tag{2.4}$$

$$\mu_k = \lambda_{j_k} = 0, \qquad k = 1, \ldots, m - r_M,$$

$$\frac{|\lambda_{i_k} - \mu_{m+k}|}{|\mu_{m+k}|} \leq \|K_S\|, \qquad k = 1, \ldots, r_N, \tag{2.5}$$

$$\mu_k = \lambda_{j_k} = 0, \qquad k = m + r_N + 1, \ldots, n,$$

Proof. Let $\widehat{H}_0$ be diagonal matrix

$$\widehat{H}_0 = \begin{bmatrix} \Lambda_1 & 0 \\ 0 & \Lambda_2 \end{bmatrix} \begin{array}{l} r_M \\ r_N \end{array} , \tag{2.6}$$

where Λ_1 and Λ_2 are defined by (2.1) and (2.2). Then we can write $\widehat{H}_0 = D^* A D$, where

$$D = \begin{bmatrix} |\Lambda_1|^{1/2} & 0 \\ 0 & |\Lambda_2|^{1/2} \end{bmatrix}, \qquad A = \begin{bmatrix} J_1 & 0 \\ 0 & J_2 \end{bmatrix}.$$

Here J_1 and J_2 are diagonal matrices with signs of eigenvalues of Λ_1 and Λ_2, respectively.

Note that $\widehat{H}$ from (2.3) can be considered as a perturbation of $\widehat{H}_0$. Indeed, since

$$\widehat{H} = D^* D^{-*} \widehat{H} D^{-1} D,$$

we have $\widehat{H} = D^*(A + \delta A)D$, where

$$\delta A = \begin{bmatrix} 0 & K_S^* \\ K_S & 0 \end{bmatrix}.$$

By a result of Veselić and Slapničar [9] we know that

$$1-\eta \leq \frac{\widetilde{\lambda}_i}{\lambda_i} \leq 1+\eta$$

where $\eta = \eta_H = \|\delta A\| \|\widehat{A}^{-1}\|$, which imply

$$\frac{|\lambda_{i_k} - \mu_{m-r_M+k}|}{|\mu_{m-r_M+k}|} \leq \|\widehat{A}^{-1}\| \|\delta A\|, \qquad k = 1, \ldots, r_M.$$

Since $\|\widehat{A}^{-1}\| \|\delta A\| = \|K_S\|$, we obtained the first part of (2.4). Similarly holds for (2.5). ∎

The bound obtained by this theorem is a proper generalization of [1, Theorem 1.1] to indefinite singular Hermitian matrices. Indeed, in the case when $J = I$ that is, for $H = GG^*$ positive semidefinite, $\|K_S\| = \sin\angle\left(\mathcal{Y}_G, \mathcal{U}_G^{\perp}\right)$, where $\mathcal{Y}_G = G^*\mathcal{X}$, $\mathcal{U}_G = G^*\mathcal{X}^{\perp}$ and angle function is defined by (see [10])

$$\sin\angle\left(\mathcal{Y}_G, \mathcal{U}_G^{\perp}\right) = \min\left\{\|P_{\mathcal{U}_G} P_{\mathcal{Y}_G}\|, \|P_{\mathcal{U}_G^{\perp}} P_{\mathcal{Y}_G^{\perp}}\|\right\},$$

where $P_{\mathcal{M}}$ is orthogonal projector onto $\mathcal{M}$. Since, for the indefinite Hermitian matrix $H = GJG^*$ we can not express $\|K_S\|$ in the terms of sine of canonical angles, in the following corollary we present the upper bound for $\|K_S\|$ which contains such a sine.

Corollary 2.2. *Let $H = GJG^*$, J, X and $X_\perp$ be as in Theorem 2.1. If $G^*X_1 = 0$ and $G^*X_{\perp,2} = 0$ and if we set $\mathcal{W}_G = \mathcal{R}(JG^*X_\perp)$ $\mathcal{Y}_G = \mathcal{R}(G^*X)$, then*

$$\|K_S\| \leq \|U\| \|Y\| \sin\phi, \tag{2.7}$$

where $U = G^*X_{\perp,1}|\Lambda_2|^{-\frac{1}{2}}$, $Y = G^*X_2|\Lambda_1|^{-\frac{1}{2}}$ and $\sin\phi$ is defined by

$$\sin\phi = \sin\angle\left(\mathcal{Y}_G, \mathcal{W}_G^{\perp}\right) = \min\left\{\|P_{\mathcal{W}_G} P_{\mathcal{Y}_G}\|, \|P_{\mathcal{W}_G^{\perp}} P_{\mathcal{Y}_G^{\perp}}\|\right\}. \tag{2.8}$$

Proof. From $K = X_{\perp,1}HX_2$ and the definition of K_S, we have

$$\begin{aligned} K_S &= |\Lambda_2|^{-\frac{1}{2}} K |\Lambda_1|^{-\frac{1}{2}} = |\Lambda_2|^{-\frac{1}{2}} X_{\perp,1}^* H X_2 |\Lambda_1|^{-\frac{1}{2}} = |\Lambda_2|^{-\frac{1}{2}} X_{\perp,1}^* GJG^* X_2 |\Lambda_1|^{-\frac{1}{2}} \\ &= \left(G^*X_{\perp,1}|\Lambda_2|^{-\frac{1}{2}}\right)^* J \left(G^*X_2|\Lambda_1|^{-\frac{1}{2}}\right) = U^*JY = W^*Y, \end{aligned}$$

where $W = JU$. Note that

$$\begin{aligned} Y^*JY &= |\Lambda_1|^{-\frac{1}{2}} X_2^* GJG^* X_2 |\Lambda_1|^{-\frac{1}{2}} \\ &= |\Lambda_1|^{-\frac{1}{2}} X_2^* H X_2 |\Lambda_1|^{-\frac{1}{2}} = |\Lambda_1|^{-\frac{1}{2}} \Lambda_1 |\Lambda_1|^{-\frac{1}{2}} = J_1, \end{aligned}$$

$$\begin{aligned} W^*JW &= |\Lambda_2|^{-\frac{1}{2}} X_{\perp,1}^* GJG^* X_{\perp,1} |\Lambda_2|^{-\frac{1}{2}} \\ &= |\Lambda_2|^{-\frac{1}{2}} X_{\perp,1}^* H X_{\perp,1} |\Lambda_2|^{-\frac{1}{2}} = |\Lambda_2|^{-\frac{1}{2}} \Lambda_2 |\Lambda_2|^{-\frac{1}{2}} = J_2. \end{aligned}$$

This shows that W and Y have J-orthogonal columns. Further, from

$$\mathcal{R}(Y) = \mathcal{R}\left(G^*X_2|\Lambda_1|^{-\frac{1}{2}}\right) = \mathcal{R}\left(G^*X_2\right) \subset \mathcal{R}\left(G^*X\right) = \mathcal{Y}_G,$$

$$\mathcal{R}(W) = \mathcal{R}\left(JG^*X_{\perp,1}|\Lambda_2|^{-\frac{1}{2}}\right) = \mathcal{R}\left(JG^*X_\perp\right) \subset \mathcal{R}\left(JG^*X\right) = \mathcal{W}_G,$$

and from $G^*X_1 = 0$ and $G^*X_{\perp,2} = 0$ it follows that $\mathcal{R}(Y) = \mathcal{Y}_G$ and $\mathcal{R}(W) = \mathcal{W}_G$. Finally, let $W = Q_W R_W$ and $Y = Q_Y R_Y$ be QR-decompositions of W and Y, respectively. Then

$$\|K_S\| = \|W^*Y\| \leq \|R_W^*\|\|R_Y\|\|Q_W^*Q_Y\|. \tag{2.9}$$

The columns of Q_W and Q_Y form orthonormal basis for $\mathcal{W}_G$ and $\mathcal{Y}_G$, respectively. Drmač and Hari have shown in proof of [1, Theorem 1.1], that $\|Q_W^*Q_Y\| = \sin\phi$. Now, using this and the fact that

$$\|R_W^*\| = \|U\|, \qquad \|R_Y\| = \|Y\|,$$

from (2.9) follows (2.7). ■

Inserting (2.7) into (2.4) and (2.5) we obtain the bound which is a proper generalization of [1, Theorem 1.1] to indefinite Hermitian matrices, since in semidefinite case ($J = I$) this bound is equal to $\sin\phi$, and $W \equiv U$ and Y have orthonormal columns.

Note that in the positive definite case the angle function $\angle(\mathcal{Y}_G, \mathcal{Z}_G)$ defined by (2.8) does not depend on G but only on H (see [1]). However, in indefinite case this is not true in general. The dependence of the angle function $\angle(\mathcal{Y}_G, \mathcal{Z}_G)$ on the factor G, where $H = GJG^*$, for a nonsingular indefinite matrix H, has been considered in [8]. Now we will present a similar result for the indefinite possible singular matrix H. Let

$$H = G_1JG_1^* = G_2JG_2^* \tag{2.10}$$

be decompositions of the matrix H, $i = 1, 2$, and let $W_i = Q_{W_i}R_{W_i}$ and $Y_i = Q_{Y_i}R_{Y_i}$ be QR decompositions of W_i and Y_i respectively, where W_i and Y_i are defined as in the proof of the Corollary 2.2 (W_i and Y_i corresponds with G_i) for $i = 1, 2$. Note that (2.9) can be written as $\|K_S\| = \|W_i^*Y_i\|$, $i = 1, 2$. Now using this equalities we can write the simple inequalities

$$\begin{aligned}\sin\phi_2 = \|R_{W_2}^{-*}R_{W_2}^*Q_{W_2}^*Q_{Y_2}R_{Y_2}R_{Y_2}^{-1}\| &\leq \|R_{W_2}^{-*}\|\,\|R_{Y_2}^{-1}\|\|K_S\| \\ &\leq \|R_{W_2}^{-*}\|\,\|R_{Y_2}^{-1}\|\,\|R_{W_1}\|\,\|R_{Y_1}\|\sin\phi_1\end{aligned}$$

and similarly

$$\sin\phi_1 \leq \|R_{W_1}^{-*}\|\,\|R_{Y_1}^{-1}\|\,\|R_{W_2}\|\,\|R_{Y_2}\|\sin\phi_2\,.$$

Now, from the above inequalities we can write the following bound

$$\|U_1^\dagger\|\,\|Y_1^\dagger\|\,\|U_2\|\,\|Y_2\| \leq \frac{\sin\phi_2}{\sin\phi_1} \leq \|U_2^\dagger\|\,\|Y_2^\dagger\|\,\|U_1\|\,\|Y_1\|\,, \tag{2.11}$$

where we have use the fact that $\|R_{W_i}^{-*}\| = \|U_i^\dagger\|$ and $\|R_{Y_i}^{-*}\| = \|Y_i^\dagger\|$, $i = 1, 2$. Here $\dagger$ denotes the generalized inverse.

Note that bound (2.11) depends on a magnitude of the numbers which are similar to the condition numbers of the matrices U_i and Y_i (condition number of the matrix U is defined as $\|U^\dagger\|\,\|U\|$). The classes of so called "well-behaved matrices" for which exist useful bounds for conditions of U_i and Y_i have been considered in [6]. This class include matrices such as *scaled diagonal dominant matrices*, *block scaled diagonally dominant (BSDD) matrices* and *quasi-definite matrices*. Details about these bounds can be found in e.g. [7, Section 3.1] and [6].

3. Quadratic residual bound. In this section we will present quadratic relative residual bound for the eigenvalues of an indefinite singular Hermitian matrix and compare it with results from classical perturbation theory.

The main result of this section is a proper generalization of Drmač and Hari's Theorem 2.1 from [1], to indefinite, possible singular Hermitian matrices.

In the following theorem $\sigma_{\min}(\cdot)$ denotes the smallest singular value of a matrix. We will use the same notation as in Theorem 2.1. For a given nonzero eigenvalue λ of H we shall choose the bases X and $X^{\perp}$ such that

$$\Lambda_1 = \Xi_\lambda \oplus \widehat{\Xi}_\lambda, \qquad \Lambda_2 = \Omega_\lambda \oplus \widehat{\Omega}_\lambda, \tag{3.1}$$

where the diagonals of Ξ_λ and Ω_λ approximate λ in the sense of Theorem 2.1.

Let Λ_1 and Λ_2 be decomposed as

$$\Lambda_1 = \begin{bmatrix} |\Xi_\lambda|^{1/2} & 0 \\ 0 & |\widehat{\Xi}_\lambda|^{1/2} \end{bmatrix} \begin{bmatrix} J_{11} & 0 \\ 0 & J_{22} \end{bmatrix} \begin{bmatrix} |\Xi_\lambda|^{1/2} & 0 \\ 0 & |\widehat{\Xi}_\lambda|^{1/2} \end{bmatrix}, \tag{3.2}$$

$$\Lambda_2 = \begin{bmatrix} |\Omega_\lambda|^{1/2} & 0 \\ 0 & |\widehat{\Omega}_\lambda|^{1/2} \end{bmatrix} \begin{bmatrix} \bar{J}_{11} & 0 \\ 0 & \bar{J}_{22} \end{bmatrix} \begin{bmatrix} |\Omega_\lambda|^{1/2} & 0 \\ 0 & |\widehat{\Omega}_\lambda|^{1/2} \end{bmatrix}, \tag{3.3}$$

where $J_{11},\ J_{22}\ \bar{J}_{11},\ \bar{J}_{22}$ are diagonal matrices with ± 1 on the diagonal. We write $J = J_{11} \oplus J_{22}$, $\bar{J} = \bar{J}_{11} \oplus \bar{J}_{22}$.

THEOREM 3.1. *Let H, $\mathcal{X}$ be as in the Theorem 2.1. Let $\lambda > 0$ be an eigenvalue of H of multiplicity $n(\lambda)$ (for $\lambda < 0$ we consider $-H$). Let the orthonormal bases of $\mathcal{X}$ and $\mathcal{X}^{\perp}$ be chosen such that (3.1) holds. Write $K_S = |\Lambda_2|^{-1/2} K |\Lambda_1|^{-1/2}$, where K is defined by (2.3). Suppose that there exist constants $\alpha > \gamma$ and $\beta > \gamma$ such that*

$$\|\lambda|\Xi_\lambda|^{-1} - J_{11}\| \le \gamma, \qquad \sigma_{\min}(\lambda|\widehat{\Xi}_\lambda|^{-1} - J_{22}) > \alpha, \tag{3.4}$$

$$\|\lambda|\Omega_\lambda|^{-1} - \bar{J}_{11}\| \le \gamma, \qquad \sigma_{\min}(\lambda|\widehat{\Omega}_\lambda|^{-1} - \bar{J}_{22}) > \beta. \tag{3.5}$$

If $\Xi_\lambda \oplus \Omega_\lambda$ is of order $n(\lambda)$ and $\|K_S\| \le \gamma < 1$, then

$$\|\lambda \Xi_\lambda^{-1} - I\| \le \frac{1}{1 - \dfrac{\|K_S\|^2}{\alpha\,\beta}} \frac{\|K_S\|^2}{\beta} \le \frac{\|U\|^2\|Y\|^2}{1 - \dfrac{\|U\|^2\|Y\|^2 \sin^2\phi}{\alpha\,\beta}} \frac{\sin^2\phi}{\beta}, \tag{3.6}$$

$$\|\lambda \Omega_\lambda^{-1} - I\| \le \frac{1}{1 - \dfrac{\|K_S\|^2}{\alpha\,\beta}} \frac{\|K_S\|^2}{\alpha} \le \frac{\|U\|^2\|Y\|^2}{1 - \dfrac{\|U\|^2\|Y\|^2 \sin^2\phi}{\alpha\,\beta}} \frac{\sin^2\phi}{\beta}, \tag{3.7}$$

where $K_S = U^* J Y$ and U, Y and $\sin\phi$ are defined as in Corollary 2.2.

Proof. Our proof is similar to the proof of [1, Theorem 2.1], and most of it can be omitted but we include the whole proof for completeness. Without loss of generality we can assume

$$H = \begin{array}{c} \begin{bmatrix} \Lambda_1 & K^* \\ K & \Lambda_2 \end{bmatrix} \end{array} \begin{array}{c} r_M \\ r_N \end{array}. \tag{3.8}$$

where Λ_1 and Λ_2 are given by (3.1). Otherwise one can work with $\widehat{H}$ from the proof of the Theorem 2.1. Matrix H is a non-singular matrix of dimension $r \times r$.

Matrices J and $\bar{J}$ are diagonal matrices which contain signs of Λ_1 and Λ_2 from (3.1). It is easy to show that under the assumptions (3.4) and (3.5), $J_{11} = I$ and $\bar{J}_{11} = I$, thus $|\Xi_\lambda| = \Xi_\lambda$ and $|\Omega_\lambda| = \Omega_\lambda$. Indeed, let us show that $J_{11} = I$. From (3.4) we have

$$\|\lambda|\Xi_\lambda|^{-1} - J_{11}\| < 1$$

or

$$\max_j \left| \frac{\lambda}{|\lambda_j|} - \operatorname{sign}(\lambda_j) \right| < 1, \quad j = 1, \ldots, \dim(\Xi_\lambda).$$

The last inequality is equivalent to

$$|\lambda - sign(\lambda_j)|\lambda_j|| < |\lambda_j|,$$

which can not be obtained for $\lambda_j < 0$, thus $\lambda_j > 0$ for all j, and we conclude that $J_{11} = I$.

By Sylvester's low of inertia, the matrix

$$H_S(\lambda) = (|\Lambda_1| \oplus |\Lambda_2|)^{-1/2}(H - \lambda I)(|\Lambda_1| \oplus |\Lambda_2|)^{-1/2}$$

has rank $n - n(\lambda)$. It has the following block structure:

$$H_S(\lambda) = \begin{bmatrix} I - \lambda|\Xi_\lambda|^{-1} & 0 & (K_S^{(1,1)})^* & (K_S^{(2,1)})^* \\ 0 & J_{22} - \lambda|\widehat{\Xi}_\lambda|^{-1} & (K_S^{(1,2)})^* & (K_S^{(2,2)})^* \\ (K_S^{(1,1)}) & (K_S^{(1,2)}) & I - \lambda|\Omega_\lambda|^{-1} & 0 \\ (K_S^{(2,1)}) & (K_S^{(2,2)}) & 0 & \bar{J}_{22} - \lambda|\widehat{\Omega}_\lambda|^{-1} \end{bmatrix}.$$

Let $\widehat{H}_S(\lambda)$ be similar matrix to $H_S(\lambda)$ defined by

$$\begin{aligned} \widehat{H}_S(\lambda) &= \Pi^T H_S(\lambda)\Pi \\ &= \begin{bmatrix} I - \lambda|\Xi_\lambda^{-1}| & (K_S^{(1,1)})^* & 0 & (K_S^{(2,1)})^* \\ (K_S^{(1,1)}) & I - \lambda|\Omega_\lambda|^{-1} & (K_S^{(1,2)}) & 0 \\ 0 & (K_S^{(1,2)})^* & J_{22} - \lambda|\widehat{\Xi}_\lambda|^{-1} & (K_S^{(2,2)})^* \\ (K_S^{(2,1)}) & 0 & (K_S^{(2,2)}) & \bar{J}_{22} - \lambda|\widehat{\Omega}_\lambda|^{-1} \end{bmatrix}. \end{aligned}$$

where Π denotes an appropriate permutation matrix.

The assumptions (3.4) and (3.5) imply

$$\sigma_{\min}((J_{22} - \lambda|\widehat{\Xi}_\lambda|^{-1}) \oplus (\bar{J}_{22} - \lambda|\widehat{\Omega}_\lambda|^{-1})) \geq \min\{\alpha, \beta\} \tag{3.9}$$

$$> \gamma \geq \|K_S\| \geq \max_{1 \leq i,j \leq 2} \|K_S^{(i,j)}\| \tag{3.10}$$

Hence the matrix

$$\begin{aligned} C &= \begin{bmatrix} J_{22} - \lambda|\widehat{\Xi}_\lambda|^{-1} & (K_S^{(2,2)})^* \\ (K_S^{(2,2)}) & \bar{J}_{22} - \lambda|\widehat{\Omega}_\lambda|^{-1} \end{bmatrix} = \begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix}, \\ C_{12} &= C_{21}^* \end{aligned}$$

and its diagonal blocks C_{11} and C_{22} are non-singular. Therefore (see [2, Section 0.7.3])

$$C^{-1} = \begin{bmatrix} \left[C_{11} - C_{12}C_{22}^{-1}C_{21}\right]^{-1} & C_{11}^{-1}C_{12}\left[C_{21}C_{11}^{-1}C_{12} - C_{22}\right]^{-1} \\ \left[C_{21}C_{11}^{-1}C_{12} - C_{22}\right]^{-1}C_{21}C_{11}^{-1} & \left[C_{22} - C_{21}C_{11}^{-1}C_{12}\right]^{-1} \end{bmatrix},$$

provided that all matrices in the brackets are non-singular. However this follows since these matrices are (signed) Schur complements of C_{11} and C_{22} in C. By the last assumption C is of order $n - n(\lambda)$ what is also the rank of $\widehat{H}_S(\lambda)$. Since C is non-singular its Schur complement in $\widehat{H}_S(\lambda)$ must be zero ([3, p.183]). Hence

$$\begin{bmatrix} I - \lambda\Xi_\lambda^{-1} & (K_S^{(1,1)})^* \\ (K_S^{(1,1)}) & I - \lambda\Omega_\lambda^{-1} \end{bmatrix} = \begin{bmatrix} 0 & (K_S^{(2,1)})^* \\ (K_S^{(2,1)}) & 0 \end{bmatrix} C^{-1} \begin{bmatrix} 0 & (K_S^{(1,2)})^* \\ (K_S^{(1,2)}) & 0 \end{bmatrix}. \quad (3.11)$$

From (3.11) we obtain

$$I - \lambda\Xi_\lambda^{-1} = (K_S^{(2,1)})^* \left[\bar{J}_{22} - \lambda|\widehat{\Omega}_\lambda|^{-1} - K_S^{(2,2)}(J_{22} - \lambda|\widehat{\Xi}_\lambda|^{-1})^{-1}(K_S^{(2,2)})^*\right]^{-1} K_S^{(2,1)}$$

$$I - \lambda\Omega_\lambda^{-1} = (K_S^{(1,2)}) \left[J_{22} - \lambda|\widehat{\Xi}_\lambda|^{-1} - (K_S^{(2,2)})^*(\bar{J}_{22} - \lambda|\widehat{\Omega}_\lambda|^{-1})^{-1}(K_S^{(2,2)})\right]^{-1} (K_S^{(1,2)})^*.$$

Now applying standard 2-norm to the expressions on the left- and right-hand side we obtain

$$\|I - \lambda\Xi_\lambda^{-1}\| \leq \frac{\|K_S^{(2,1)}\|^2}{\beta - \frac{\|K_S^{(2,2)}\|^2}{\alpha}}$$

$$\|I - \lambda\Omega_\lambda^{-1}\| \leq \frac{\|K_S^{(1,2)}\|^2}{\alpha - \frac{\|K_S^{(2,2)}\|^2}{\beta}}$$

Since

$$\max_{1\leq i,j\leq 2} \|K_S^{(i,j)}\| \leq \|K_S\|$$

the first inequalities of (3.6) and (3.7) are proved. The upper bounds for (3.6) and (3.7) follow from Corollary 2.2. ■

Theorem 3.1 is a proper generalization of [1, Theorem 2.1], since in positive semidefinite case $J = I$ and bounds (3.6) and (3.7) have the same form as the bound from [1, Theorem 2.1].

The following example is indefinite version of Example 2.4 from [1], and it shows that for a certain Hermitian matrices results from Theorem 1.2 and Theorem 1.3 can not be applicable.

Example 1. Let

$$H = \begin{bmatrix} -10^{10} & 1 & 10^{-13} \\ 1 & -2\cdot 10^{-5} & 10^{-7} \\ 10^{-13} & 10^{-7} & -10^{-5} \end{bmatrix}, \quad X = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \quad X^\perp = \begin{bmatrix} 0 & 1 \\ 0 & 0 \\ 1 & 0 \end{bmatrix}.$$

Then

$$M = \begin{bmatrix} 2\cdot 10^{-5} \end{bmatrix}, \quad R = \begin{bmatrix} 1 & 0 & 10^{-7} \end{bmatrix}^*, \quad \|R\| \approx 1.$$

The separation δ from Theorem 1.2 is of order 10^{-5}, and this also holds for separations δ_k and $\widetilde{\delta}_k$ from Theorem 1.3. In Theorem 1.3 $\|R_0\| = \|R\|$. All this means that in this these theorems are not applicable.

On the other hand Theorem 2.1 ensure that for some $j_0 \in \{1,2,3\}$ holds

$$\frac{\lambda_{j_0} - 2 \cdot 10^{-5}}{\sqrt{|\lambda_{j_0}| \cdot 2 \cdot 10^{-5}}} < 1.24 \cdot 10^{-2}. \tag{3.12}$$

Since $\|K_s\| \approx 7.4 \cdot 10^{-3}$, we can take $\gamma = 2 \cdot 10^{-2}$ in Theorem 2.1. If we consider (3.12) and by taking $\beta = 0.9$ we can assume that conditions from (3.4) are satisfied. Now, the bound from Theorem 3.1 yields

$$\frac{\lambda_{j_0} - 2 \cdot 10^{-5}}{\lambda_{j_0}} < 6.1 \cdot 10^{-5}.$$

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