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# Relative Perturbation Theory for Matrix Spectral Decompositions

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to Branka

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# Contents

# Notation

| | |
|---|---|
| $A^*$ | The conjugate transpose of $A$ |
| $I$ $(I_n)$ | The identity matrix (of order $n$) |
| $\|\cdot\|$ | The 2-norm |
| $\|\cdot\|_F$ | The Frobenius norm |
| $A \oplus B$ | The direct sum of $A$ and $B$ |
| $\mathbf{C}$ | The set of complex numbers |
| $\mathbf{C}^n$ | The set of complex $n$-dimensional vectors |
| $\mathbf{C}^{m\times n}$ | The set of complex $m \times n$-dimensional matrices |
| $\mathbf{\vert} A \mathbf{\vert}$ | The spectral absolute value of $A$ |
| $\vert A \vert$ | The matrix whose elements are the absolute values of the elements of $A$ |
| $\mathcal{R}(A)$ | The column space of $A$ |
| $\mathcal{R}(A)^{\perp}, B_{\perp}$ | The orthogonal complement of $\mathcal{R}(A)$, $B$ |
| $\angle(x, y)$ | The angle between $x$ and $y$ |
| $\sqrt{A}$ | The matrix square root of $A$ |
| $\sigma(A)$ | The set of eigenvalues of $A$ |
| $A_{ij}$ | The $(i, j)$-th element of $A$ |
| $\operatorname{diag}(a_1, \ldots, a_n)$ | The diagonal matrix with $a_1, \ldots, a_n$ on diagonal |
| $\operatorname{tr}(A)$ | The trace of $A$ |

# Chapter 1

# Introduction

The main subject of this thesis is *relative perturbation theory* for various eigenvalue decompositions or spectral decompositions. We consider following eigenvalue problems: the Hermitian eigenvalue problem, the hyperbolic eigenvalue problem, and the generalized eigenvalue problem.

The Hermitian eigenvalue problem is given by

$$Hu_i \quad = \quad \lambda_i u_i, \tag{1.1}$$

or

$$H = U\Lambda U^* = \sum_{i=1}^{n} \lambda_i u_i u_i^*,$$

where $H$ is a Hermitian matrix of order $n$, $\Lambda = \operatorname{diag}(\lambda_i)$ is a diagonal matrix whose diagonal elements are the eigenvalues of $H$, and $U = [\,u_1 \quad u_2 \quad \cdots \quad u_n\,]$ is unitary matrix whose $i$-th column is the eigenvector which corresponds to $\lambda_i$. Further, subspace $\mathcal{U}$ is an invariant subspace of matrix $H$ if

$$H\mathcal{U} \subseteq \mathcal{U}.$$

Since $H$ is Hermitian it has a system of orthonormal eigenvectors $u_1, u_2, \ldots, u_n$ that span $\mathbf{C}^n$. Thus, if subspace $\mathcal{U}$ is spanned by vectors $u_1, \ldots, u_k$ we say that subspace $\mathcal{U}$ belongs to the eigenvalues $\lambda_1, \ldots, \lambda_k$.

We present a relative variant of Hoffman and Wielandt bound [46, Theorem IV.3.1] for eigenvalues of the problem (1.1).

**Theorem 1.1 (Hoffman-Wielandt)** *Let $A$ and $\tilde{A}$ be normal. Then*

$$\min_{\pi} \sqrt{\sum_i |\tilde{\lambda}_{\pi(i)} - \lambda_i|^2} \leq \|\tilde{A} - A\|_F,$$

*where* $\pi$ *ranges over all permutations of integers* $1, 2, \ldots, n$.

If $A$, in the above theorem, is the Hermitian matrix then permutation $\pi$ is identity.

Further, we present a relative variant, of the well-known $\sin \Theta$ theorem by Davis and Kahan [4], [46, Section V.3.3] for simple invariant subspaces $\mathcal{U}$. A simple invariant subspace is defined as follows: if $\{\lambda_{i_1}, \ldots \lambda_{i_k}\}$ and $\{\lambda_{i_{k+1}}, \ldots \lambda_{i_n}\}$ are two sets which do not contain common eigenvalues, then eigenvectors $u_{i_1}, \ldots, u_{i_k}$ and $u_{i_{k+1}}, \ldots, u_{i_n}$ which correspond to these sets, respectively, span simple invariant subspaces according to [46, Definition V.1.2.].

Further, we consider the hyperbolic eigenvalue problem

$$Hx_i = \lambda_i J x_i, \qquad i = 1, 2, \cdots, n, \tag{1.2}$$

where $H$ is a $n \times n$ Hermitian positive definite or positive semidefinite matrix, and $J$ is a non-singular matrix of the form $J = \operatorname{diag}(\pm 1)$. For the problem (1.2), we also present the relative version of Hoffman-Wielandt-type of bound for the eigenvalues of the pair $(H, J)$, and relative variant of $\sin \Theta$ theorem for simple invariant subspaces.

In both cases we consider two kinds of perturbations of Hermitian matrix $H$.

1. The first type of perturbation is when Hermitian matrix $H$ is given in graded form, that is,
$$H = D^* A D, \tag{1.3}$$
where $D$ and $A$ are non-singular, and is perturbed to
$$\tilde{H} = H + \delta H = D^*(A + \delta A)D.$$
We call such perturbation additive.

2. Second type of perturbation is when a Hermitian matrix $H$ is perturbed to
$$\tilde{H} = H + \delta H = (I + E)^* H (I + E).$$
We call such perturbation multiplicative.

We also give relative linear and quadratic residual bounds for eigenvalues of an indefinite, possibly singular, Hermitian matrix $H$ given in factorized form $H = GJG^*$, where $G$ has a full column rank and $J$ is diagonal and non-singular with $\pm 1$ on the diagonal. Finally, we give the relative version of Hoffman-Wielandt type bound for the generalized eigenvalue problem of the pair $(H, M)$, where $H$ is positive semidefinite and $M$ is non-singular. We also present the relative variant of $\sin \Theta$ theorem for eigensubspace of the pair $(H, M)$.

Here, our perturbation bounds are given for the case when the pair $(H, M)$ is perturbed to $(H + \delta H, M + \delta M)$.

## Why *relative perturbation theory*?

Here we briefly describe the essential motivation for the relative perturbation theory which has been very active area of research in the past years [7, 1, 8, 55, 41, 15, 11, 30, 29, 16, 24] (see also the review article [22]). Suppose that we have to compute an eigenvalue of a Hermitian matrix $H$. Then a number $\tilde{\lambda}$, which is produced by some numerical software, is not the desired eigenvalue. If this software is reliable, then there is a round-off error analysis which assures that $\tilde{\lambda}$ is an eigenvalue of a nearby matrix $H + E$, where $E$ is small in some sense. Now, we can take perturbation theory to estimate the absolute error in $\tilde{\lambda}$. For instance, the well-known Weyl theorem bounds the absolute distance between $\tilde{\lambda}$ and the closest eigenvalue $\lambda$ of $H$ by

$$|\tilde{\lambda} - \lambda| \leq \|E\|. \tag{1.4}$$

The bound (1.4) bounds absolute error. Thus, we call bounds of this form absolute. Sometimes the absolute error bounds are useless, especially when we are interested in small eigenvalues. In such a case, it is better to use relative error bound. For instance, for non-zero eigenvalue $\lambda$, simply by dividing (1.4) by $\lambda$, we obtain the relative error bound

$$\frac{|\tilde{\lambda} - \lambda|}{|\lambda|} \leq \frac{\|E\|}{|\lambda|}. \tag{1.5}$$

The relative error bound (1.5), can sometimes be too pessimistic for tiny eigenvalues. The following example illustrates that fact.

**Example 1.1** [22] *Let $H$ be diagonal matrix*

$$H = \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix},$$

*where $\lambda_1 \geq \ldots \geq \lambda_l > 0 > \lambda_{l+1} \geq \ldots \geq \lambda_n$. Storing of $H$ in floating point arithmetic produces a perturbed matrix*

$$H + E = \begin{bmatrix} \lambda_1(1+\varepsilon_1) & & \\ & \ddots & \\ & & \lambda_n(1+\varepsilon_n) \end{bmatrix},$$

*where $\lambda_i(1+\varepsilon_i)$ has the same sign as $\lambda_i$ $i = 1,\ldots,n$, and $|\varepsilon_i| \le \varepsilon$ and $\varepsilon > 0$ reflects the machine accuracy. The error in eigenvalue $\tilde{\lambda}_i$ of $H + E$ is bounded by (1.4) as*

$$\min_i |\lambda_i - \tilde{\lambda}_i| \le \|E\| = \max_k |\lambda_k \varepsilon_k| \le \varepsilon \max_k |\lambda_k|.$$

*This bound is realistic for absolutely large eigenvalues. For example if $|\tilde{\lambda}_i|$ is closest to $|\lambda_{\max}|$, where $\lambda_{\max} = \max\{|\lambda_1|,\ldots,|\lambda_n|\}$, then*

$$\frac{|\lambda_{\max} - |\tilde{\lambda}_i||}{|\lambda_{\max}|} \le \varepsilon.$$

*Since the relative error in all eigenvalues does not exceed $\varepsilon$, the bound is tight in this case.*

*However the bound is too pessimistic for tiny eigenvalues. Indeed, if $|\tilde{\lambda}_i|$ is closest to an eigenvalue $\lambda_{\min}$, where $\lambda_{\min} = \min\{|\lambda_1|,\ldots,|\lambda_n|\}$, then*

$$\frac{|\lambda_{\min} - |\tilde{\lambda}_i||}{|\lambda_{\min}|} \le \varepsilon \frac{|\lambda_{\max}|}{|\lambda_{\min}|}.$$

*The bound is much more larger than $\varepsilon$, when the magnitude of the eigenvalues varies widely. This example shows that bound is not tight, since the relative error does not exceed $\varepsilon$.*

*However, note that $H$ can be decomposed as $H = DAD$, where*

$$D = \mathrm{diag}\,(\sqrt{|\lambda_1|},\ldots,\sqrt{|\lambda_n|}) \quad \text{and} \quad A = I_l \oplus (-I_{n-l}),$$

*and $\tilde{H} = D(A + \delta A)D$, where*

$$\delta A = \mathrm{diag}\,(\varepsilon_1,\ldots,\varepsilon_l,-\varepsilon_{l+1},\ldots,-\varepsilon_n).$$

*Thus Theorem 3.1 from Table 1.1 gives*

$$\sqrt{\sum_{i=1}^{n} \frac{|\tilde{\lambda}_i - \lambda_i|^2}{|\tilde{\lambda}_i\,\lambda_i|}} \;\le\; \frac{\sqrt{\varepsilon_1^2 + \cdots + \varepsilon_n^2}}{\sqrt{1 - \max_i |\varepsilon_i|}}.$$

*Note that here $V = I$ and $\tilde{V} = I$ since $H$ and $\tilde{H}$ are diagonal.*

As we can see from the above example, relative perturbation bounds can be much more accurate then bounds obtained using classical approach, especially for the tiny eigenvalues.

Main results this thesis are schematically given by Tables 1.1- 1.4 for eigenvalues, and by Tables 1.5 - 1.7 for eigenvector perturbation bounds.

Table 1.1: Eigenvalue perturbations – relative perturbation bounds for the Hermitian eigenvalue problem

| $H$ | $\tilde{H}$ | THEOREM |
|---|---|---|
| $H = D^*AD$ indefinite non − singular | $\tilde{H} = D^*\tilde{A}D$ $\tilde{A} = A + \delta A$ non − singular | $\sqrt{\sum_{i=1}^{n} \frac{\lvert\tilde{\lambda}_i - \lambda_i\rvert^2}{\lvert\tilde{\lambda}_i \lambda_i\rvert}} \leq \lVert V\rVert\, \lVert\tilde{V}\rVert \frac{\lVert A^{-1}\rVert \lVert\delta A\rVert_F}{\sqrt{1 - \lVert A^{-1}\rVert \lVert\delta A\rVert}}$ Theorem 3.1 |
| $H = GJG^*$ indefinite non − singular | $\tilde{H} = \tilde{G}J\tilde{G}^*$ $\tilde{G} = (I + E)^*G$ | $\sqrt{\sum_{i=1}^{n} \frac{\lvert\tilde{\lambda}_i - \lambda_i\rvert^2}{\lvert\tilde{\lambda}_i \lambda_i\rvert}} \leq \psi_M \left(\frac{1}{2}\psi_M + \sqrt{1 + \frac{1}{4}\psi_M^2}\right)$ $\psi_M = \frac{2 \cdot \lVert E\rVert_F - \lVert E\rVert \lVert E\rVert_F}{1 - \lVert E\rVert}$ Theorem 3.2 |

Table 1.1 contains the relative perturbation bounds (Section 3.1) for eigenvalues for the eigenvalue problem

$$Hu = \lambda u,$$

where $H$ is a non-singular Hermitian matrix as in (1.3). Let

$$\tilde{H}\tilde{u} = \tilde{\lambda}\tilde{u},$$

be corresponding perturbed problem, where $\tilde{H} = D^*(A + \delta A)D$. Let

$$\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n \quad \text{and} \quad \tilde{\lambda}_1 \geq \tilde{\lambda}_2 \ldots \geq \tilde{\lambda}_n$$

be the eigenvalues of $H$ and $\tilde{H}$,respectively. Further, let $A = Q|\Delta|^{1/2}J|\Delta|^{1/2}Q^*$, be eigenvalue decomposition of $A$, where $Q$ is unitary, $|\Delta|$ is diagonal with positive diagonal elements, and $J$ is diagonal with $\pm 1$. Now, we can write $H = GJG^*$, where $G = D^*Q|\Delta|^{1/2}$ and $H + \delta H = \tilde{G}J\tilde{G}^*$, where $\tilde{G} = D^*Q|\Delta|^{1/2}(I + JN)^{1/2}$, where $N = |\Delta|^{-1/2}Q^*\delta AQ|\Delta|^{-1/2}$. In Table 1.1 matrices $V = \begin{bmatrix} V_1 & V_2 \end{bmatrix}$ and $\tilde{V} = \begin{bmatrix} \tilde{V}_1 & \tilde{V}_2 \end{bmatrix}$ denote non-singular $J$-unitary matrices which simultaneously diagonalize the pairs $(G^*G, J)$ and $(\tilde{G}^*\tilde{G}, J)$, respectively, that is

$$V^*G^*GV = |\Lambda|, \quad V^*JV = J, \qquad \tilde{V}^*\tilde{G}^*\tilde{G}\tilde{V} = |\tilde{\Lambda}| \quad \tilde{V}^*J\tilde{V} = J.$$

Table 1.2: Eigenvalue perturbations – relative residual bounds for the Hermitian eigenvalue problem

<table>
<tr><th>MATRIX</th><th>LINEAR RESIDUAL BOUND</th></tr>
<tr><td>$H = GJG^*$<br>Hermitian<br>indefinite nonsingular<br>$U$ orthogonal</td><td>$$\begin{aligned} \max_{1\le k\le n} \frac{|\lambda_{i_k} - \mu_k|}{\mu_k} &\le \|V\|^2 \frac{\sin\psi}{1-\sin\psi} \\ \psi &= \angle(\mathcal{R}(G^*U), \mathcal{R}(G^{-1}U)) \end{aligned}$$<br>Theorem 4.1</td></tr>
<tr><td>$H = GJG^*$<br>Hermitian<br>indefinite singular<br>$U$ orthogonal<br>$J = J_1 \oplus J_2$</td><td>$$\begin{aligned} \max_{1\le k\le n} \frac{|\lambda_{i_k} - \mu_k|}{\mu_k} &\le \|K_S\| \\ \|K_S\| &\le \|U\|\|Y\| \sin\phi, \\ \phi &= \angle(\mathcal{R}(G^*U), \mathcal{R}(G^*U^\perp)^\perp) \\ U^*JU = J_1 & \quad Y^*JY = J_2 \end{aligned}$$<br>Theorem 4.3</td></tr>
<tr><th>MATRIX</th><th>QUADRATIC RESIDUAL BOUND</th></tr>
<tr><td>$H = GJG^*$<br>Hermitian indefinite<br>possibl singular<br>$J = J_1 \oplus J_2$</td><td>$$\begin{aligned} \max_{1\le k\le n} \frac{|\lambda_{i_k} - \mu_k|}{\mu_k} &\le \frac{1}{1 - \frac{\|K_S\|}{\alpha\beta}} \frac{\|K_S\|^2}{\beta} \\ \|K_S\| &\le \|U\|\|Y\| \sin\phi, \\ \phi &= \angle(\mathcal{R}(G^*U), \mathcal{R}(G^*U^\perp)^\perp) \\ U^*JU = J_1 & \quad Y^*JY = J_2 \end{aligned}$$<br>Theorem 4.7</td></tr>
</table>

Table 1.2 contains relative linear and quadratic residual bounds (Section 4.1 and Section 4.2) for the Hermitian eigenvalue problem.

Table 1.3: Eigenvalue perturbations – relative perturbation bounds for the hyperbolic eigenvalue problem

| $(H,J)$ | $(\tilde{H},J)$ | THEOREM |
|---|---|---|
| $H=D^*AD$ positive semidefinite | $\tilde{H}=D^*\tilde{A}D$ $\tilde{A}=A+\delta A$ positive semidefinite | $\sqrt{\sum_{i=1}^{n}\frac{\lvert\tilde{\lambda}_i-\lambda_i\rvert^2}{\lvert\tilde{\lambda}_i\,\lambda_i\rvert}} \leq \psi\left(\frac{1}{2}\psi+\sqrt{1+\frac{1}{4}\psi^2}\right)$ $\psi = \frac{\lVert A^{-1/2}\delta AA^{-1/2}\rVert_F}{\sqrt{1-\lVert A^{-1/2}\delta AA^{-1/2}\rVert}}$ Theorem 5.2 |
| $H=G^*G$ positive semidefinite | $\tilde{H}=\tilde{G}^*\tilde{G}$ $\tilde{G}=G(I+E)$ | $\sqrt{\sum_{i=1}^{n}\frac{\lvert\tilde{\lambda}_i-\lambda_i\rvert^2}{\lvert\tilde{\lambda}_i\,\lambda_i\rvert}} \leq \lVert X_1\rVert\lVert\tilde{X}_1\rVert\cdot\frac{2\cdot\lVert E\rVert_F-\lVert E\rVert_F\lVert E\rVert}{1-\lVert E\rVert}$ Theorem 5.4 |

Table 1.3 contains the relative perturbation bounds (Section 5.1) for the eigenvalues of the hyperbolic eigenvalue problem

$$Hx \;=\; \lambda Jx,$$

where $H$ is a positive semidefinite and $J$ is diagonal with $\pm 1$ on its diagonal, and

$$\lambda_1\geq\lambda_2\geq\ldots\geq\lambda_n \quad\text{and}\quad \tilde{\lambda}_1\geq\tilde{\lambda}_2\ldots\geq\tilde{\lambda}_n$$

are eigenvalues of the pairs $(H,J)$ and $(\tilde{H},J)$, respectively. The matrices $X=\begin{bmatrix} X_1 & X_2\end{bmatrix}$ and $\tilde{X}=\begin{bmatrix}\tilde{X}_1 & \tilde{X}_2\end{bmatrix}$ in the Table 1.3 denote non-singular $J$-unitary matrices which simultaneously diagonalize the pairs $(H,J)$ and $(\tilde{H},J)$, respectively, that is,

$$\begin{bmatrix} X_1^* \\ X_2^*\end{bmatrix} H\,[\,X_1 \quad X_2\,] = \begin{bmatrix} \lvert\Lambda_1\rvert & \\ & \lvert\Lambda_2\rvert\end{bmatrix}, \qquad \begin{bmatrix} X_1^* \\ X_2^*\end{bmatrix} J\,[\,X_1 \quad X_2\,] = \begin{bmatrix} J_1 & \\ & J_2\end{bmatrix}$$

$$\begin{bmatrix} \tilde{X}_1^* \\ \tilde{X}_2^*\end{bmatrix} \tilde{H}\,[\,\tilde{X}_1 \quad \tilde{X}_2\,] = \begin{bmatrix} \lvert\tilde{\Lambda}_1\rvert & \\ & \lvert\tilde{\Lambda}_2\rvert\end{bmatrix}, \qquad \begin{bmatrix} \tilde{X}_1^* \\ \tilde{X}_2^*\end{bmatrix} J\,[\,\tilde{X}_1 \quad \tilde{X}_2\,] = \begin{bmatrix} J_1 & \\ & J_2\end{bmatrix}.$$

Table 1.4: Eigenvalue perturbations – relative perturbation bounds for generalized eigenvalue problem

| $(H, M)$ | $(\tilde{H}, \tilde{M})$ | THEOREM |
|---|---|---|
| $H = D^*AD$ positive semidefinite<br><br>$M = LJL^*$ $\mathrm{non-singular}$ | $\tilde{H} = D^*(A + \delta A)D$ positive semidefinite<br><br>$\tilde{M} = \tilde{L}J\tilde{L}^*$ $\mathrm{non-singular}$ | $\sqrt{\sum_{i=1}^{r} \left( \frac{\lvert\lambda_i - \tilde{\lambda}\tau(i)\rvert}{\sqrt{\lvert\tilde{\lambda}_{\tau(i)}\rvert} \max\{\sqrt{\lvert\lambda_i\rvert}, \sqrt{\lvert\lambda_{\tau(i)}\rvert}\}} \right)^2} \leq$ $\frac{\lVert F\rVert\, \lVert\tilde{F}\rVert}{1 - \lVert\delta L L^{-1}\rVert} \cdot$ $\cdot \left[ \frac{\lVert A^{-1/2}\delta A A^{-1/2}\rVert_F}{\sqrt{1 - \lVert A^{-1/2}\delta A A^{-1/2}\rVert}} + \lVert F\rVert\, \lVert\tilde{F}\rVert \frac{\lVert L^{-*}\delta M L^{-1}\rVert_F^2}{1 - \lVert\delta L L^{-1}\rVert} \right]$<br><br>Theorem 6.4 |

Let

$$Hx \quad = \quad \lambda M x, \tag{1.6}$$

be generalized eigenvalue problem, where $H = D^*AD$ where $A$ is positive definite, $D$ has a full column rank, and $M$ is non-singular. Let

$$\tilde{H}\tilde{x} = \tilde{\lambda}\tilde{M}\tilde{x},$$

be corresponding perturbed problem. Table 1.4 contains relative perturbation bounds (Section 6.1) for eigenvalues of the pair $(H, M)$, where $F$ and $\tilde{F}$ are $J$-unitary matrices which simultaneously diagonalize the pairs $(L^{-*}D^*ADL^{-1}, J)$ and $((I + \delta LL^{-1})^{-*}L^{-*}D^*(A + \delta A)DL^{-1}(I + \delta LL^{-1})^{-1}, J)$, respectively.

Further, let

$$H = U\Lambda U^* \quad = \quad \begin{bmatrix} U_1 & U_2 \end{bmatrix} \begin{bmatrix} \Lambda_1 & 0 \\ 0 & \Lambda_2 \end{bmatrix} \begin{bmatrix} U_1^* \\ U_2^* \end{bmatrix},$$

be eigenvalue decomposition of $H$, where

$$\Lambda_1 = \mathrm{diag}\,(\lambda_{i_1}, \lambda_{i_2}, \cdots, \lambda_{i_k}), \quad \Lambda_2 = \mathrm{diag}\,(\lambda_{j_1}, \lambda_{j_2}, \cdots, \lambda_{j_l}).$$

We assume the similar notation for perturbed quantities.

Table 1.5: Eigenvector perturbations – perturbation bounds for the Hermitian eigenvalue problem

| $H$ | $\tilde{H}$ | THEOREM |
|---|---|---|
| $H = D^*AD$ indefinite non − singular | $\tilde{H} = D^*\tilde{A}D$ $\tilde{A} = A + \delta A$ non − singular | $\|\sin\Theta\|_F \leq \frac{\|A^{-1}\|\|\delta A\|_F}{\sqrt{1-\|A^{-1}\|\|\delta A\|}} \cdot \frac{\|V_1\|\|\tilde{V}_2\|}{\mathrm{rg}(\Lambda_1,\tilde{\Lambda}_2)}$ $\Theta = \Theta(U_1, \tilde{U}_1)$ Theorem 3.3 |
| $H = GJG^*$ indefinite non − singular | $\tilde{H} = \tilde{G}J\tilde{G}^*$ $\tilde{G} = (I+E)^*G$ | $\|\sin\Theta\|_F \leq \left(\frac{\|E\|_F}{1-\|E\|} + \|E\|_F\right) \cdot \frac{1}{\mathrm{RelGap}(\Lambda_1,\tilde{\Lambda}_2)}$ $\Theta = \Theta(U_1, \tilde{U}_1)$ Theorem 3.4 |

Table 1.5 contains two bounds (Section 3.2) for $\|\sin\Theta\|_F$ where $\Theta \equiv \Theta(U_1, \tilde{U}_1)$ denotes diagonal matrix with canonical angles between $\mathcal{R}(\tilde{U}_1)$ and $\mathcal{R}(U_1)$ on diagonal. In Table 1.5 matrices $V = \begin{bmatrix} V_1 & V_2 \end{bmatrix}$ and $\tilde{V} = \begin{bmatrix} \tilde{V}_1 & \tilde{V}_2 \end{bmatrix}$ are defined similarly as in Table 1.1, where $V_1$ and $V_2$ have equal dimension as $U_1$ and $U_2$, respectively.

Table 1.6 contains two bounds (Section 5.2) for $\|\sin\Phi\|_F$, where $\Phi \equiv \Phi(X_1, \tilde{X}_1)$ denotes diagonal matrix with canonical angles between $\mathcal{R}(\tilde{X}_1)$ and $\mathcal{R}(X_1)$ on the diagonal. Here $X = \begin{bmatrix} X_1 & X_2 \end{bmatrix}$ and $\tilde{X} = \begin{bmatrix} \tilde{X}_1 & \tilde{X}_2 \end{bmatrix}$ are non-singular $J$-unitary matrices defined similarly as in Table 1.3.

Table 1.7 contains the bound (Section 6.1) for $\|\sin\Theta\|_F$ where $\Theta \equiv \Theta(\mathcal{R}(X_{11}), \mathcal{R}(\tilde{X}_{11}))$ denotes diagonal matrix with canonical angles between $\mathcal{R}(\tilde{X}_{11})$ and $\mathcal{R}(X_{11})$ on the diagonal. Here $X = [\, X_{11} \quad X_{12} \quad X_0 \,]$ is non-singular matrix, which simultaneously diagonalizes the pair $(H, M)$, that is

$$\begin{bmatrix} X_{11}^* \\ X_{12}^* \\ X_0^* \end{bmatrix} H \,[\, X_{11} \quad X_{12} \quad X_0 \,] \;=\; \begin{bmatrix} |\Lambda_{11}| & 0 & 0 \\ 0 & |\Lambda_{22}| & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

Table 1.6: Eigenvector perturbations – perturbation bounds for the hyperbolic eigenvalue problem

| $(H,J)$ | $(\tilde{H},J)$ | THEOREM |
|---|---|---|
| $H=D^*AD$ positive semidefinite | $H=D^*(A+\delta A)D$ positive semidefinite | $\Vert\sin\Phi\Vert_F \le \Vert F\Vert^2 \left(\frac{\psi}{2}+\sqrt{1+\frac{\psi^2}{4}}\right)\frac{\psi}{\mathrm{rg}(\tilde{\Lambda}_1,\Lambda_2)}$ <br> $\psi=\frac{\Vert A^{-1/2}\delta A A^{-1/2}\Vert_F}{\sqrt{1-\Vert A^{-1/2}\delta A A^{-1/2}\Vert}}$ <br> $\Phi=\Phi(\mathcal{R}(F_1),\mathcal{R}(\tilde{F}_1))$ Theorem 5.3 |
| $H=G^*G$ positive semidefinite | $H=\tilde{G}^*\tilde{G}$ <br> $\tilde{G}=G(I+E)$ | $\Vert\sin\Phi(F_1,\tilde{F}_1)\Vert_F \le \frac{2\Vert E\Vert_F-\Vert E\Vert\Vert E\Vert_F}{1-\Vert E\Vert}\cdot\frac{\Vert X\Vert\Vert\tilde{X}\Vert}{\mathrm{rg}(\tilde{\Lambda}_1,\Lambda_2)}$ <br> $\Phi=\Phi(\mathcal{R}(F_1),\mathcal{R}(\tilde{F}_1))$ Theorem 5.5 |

Table 1.7: Eigenvector perturbations – perturbation bounds for generalized eigenvalue problem

| $(H,M)$ | $(\tilde{H},\tilde{M})$ | THEOREM |
|---|---|---|
| $H=D^*AD$ positive semidefinite <br><br> $M=LJL^*$ <br> non $-$ singular | $\tilde{H}=D^*(A+\delta A)D$ positive semidefinite <br><br> $\tilde{M}=\tilde{L}J\tilde{L}^*$ <br> non $-$ singular | $\Vert\sin\Theta(\mathcal{X}_{11},\tilde{\mathcal{X}}_{11})\Vert_F \le$ <br> $\Vert F\Vert\Vert\tilde{F}\Vert\Vert L\Vert^2(1+\Vert\delta L L^{-1}\Vert)$ <br> $\cdot\left(\frac{\Psi}{Relgap}+\frac{\Vert F\Vert\,\Vert\tilde{F}\Vert}{rg}\frac{\Vert L^{-*}\delta M L^{-1}\Vert_F}{1-\Vert\delta L L^{-1}\Vert}\right)$ <br><br> Theorem 6.6 |

$$\begin{bmatrix} X_{11}^* \\ X_{12}^* \\ X_0^* \end{bmatrix} M \begin{bmatrix} X_{11} & X_{12} & X_0 \end{bmatrix} = \begin{bmatrix} J_{11} & 0 & 0 \\ 0 & J_{22} & 0 \\ 0 & 0 & J_2 \end{bmatrix} = \widehat{J},$$

where $\widehat{J}$ is diagonal and it is a possible permutation of $J$. Similarly, for perturbed quantities. In table 1.7 $F$ and $\tilde{F}$ are $J$-unitary matrices which simultaneously diagonalizes the pairs $(L^{-*}D^*ADL^{-1}, J)$ and $((I+\delta LL^{-1})^{-*}L^{-*}D^*(A+\delta A)DL^{-1}(I+\delta LL^{-1})^{-1}, J)$, respectively.

The thesis is organized as follows: in the rest of this chapter we state some of the existing relative perturbation results. In Chapter 2 we present CS-decomposition of a $J$-unitary matrix and some application of this decomposition to $J$-unitary matrices. In the last section of Chapter 2 we present two perturbation bounds for $\|F\|$, where $F$ is $J$-unitary. In Chapter 3 we present the relative version of Hoffman-Wielandt-type of bound for Hermitian eigenvalue problem and relative variant of the well-known $\sin\Theta$ theorem by Davis and Kahan for simple invariant subspaces. Further, Chapter 4 contains relative linear and quadratic residual bounds for the Hermitian eigenvalue problem. In Chapter 5 we present the relative version of Hoffman-Wielandt-type of bound for the eigenvalues, and relative variant of $\sin\Theta$ theorem for simple invariant subspaces for the hyperbolic eigenvalue problem. Finally, in Chapter 6 we present eigenvalue perturbation bound and perturbation bound for eigensubspaces for generalized eigenvalue problem.

## 1.1 Some existing relative perturbation results

There is a great number of relative perturbation bounds which deals with additive and multiplicative perturbations of diagonalisable matrices, normal matrices and Hermitian matrices. The review article [22] describes a majority of these bounds, thus here we point out only these bounds which have influence on our work.

The following theorem contains a very general result and includes a several important classes of perturbations.

**Theorem 1.2 (Veselić–Slapničar)** *Let $H$, $K$ be Hermitian and $K$ positive definite. Set $K = ZZ^*$ and*

$$|H|_K = Z|Z^{-1}HZ^{-*}|Z^*, \tag{1.7}$$

*where $|H| = \sqrt{H^2} = U|\Lambda|U^*$ is a spectral absolute value of $H$ (that is, $|H|$ is the positive definite polar factor of $H$). $|H|_K$ is independent of the freedom of choice in $Z$ (for $H$ positive definite we obviously have $|H|_K = H$). Let $\delta H$, $\delta K$ be Hermitian perturbations such that for all $x \in \mathbf{C}^n$*

$$|x^*\delta Hx| \le \eta_H x^*|H|x, \quad |x^*\delta Kx| \le \eta_K x^*Kx, \quad \forall x,\ \eta_H, \eta_K < 1, \tag{1.8}$$

*holds. Let $\lambda_i$ and $\tilde{\lambda}_i$ be the non-increasingly ordered eigenvalues of the matrix pairs $(H, K)$ and $\tilde{H} \equiv H + \delta H, \tilde{K} \equiv K + \delta K$, respectively. Then $\tilde{\lambda}_i = 0$ if and only if $\lambda_i = 0$, and for non-vanishing $\lambda_i's$ we have*

$$\frac{1-\eta_H}{1+\eta_K} \leq \frac{\tilde{\lambda}_i}{\lambda_i} \leq \frac{1+\eta_H}{1-\eta_K}. \tag{1.9}$$

**Proof**. See [55] Theorem 2.1. pp. 86-87. ∎

This inequality (1.9) implies that the perturbations which satisfy (1.8) are inertia preserving.

Note that if $K = I$ then $|H|_K = |H| = \sqrt{H^2}$, that is $|H|$ is the positive definite polar factor of $H$. Later Eisenstat and Ipsen in [17, Corollary 4.4] present the relative perturbation bounds for eigenvalues of normal matrix $A$, using definite polar factor of $A$, and Dopico, Moro and Molera in [10] use different approach to obtain relative eigenvalue perturbation bound (similar to (1.9)) for Hermitian matrix $H$. Their bounds also contain positive definite polar factor of $H$.

From Theorem 1.2 one can derive several results for additive perturbations of a single Hermitian matrix $H$. The generalized eigenvalue problem 1.6 is equivalent to the perturbations of the pair $(H, I)$ into $H + \delta H, I$, thus here we have $K = I$ and $\delta K = 0$, that is, $\eta_K = 0$.

**Corollary 1.1 (Veselić–Slapničar)** *Let $H$ be Hermitian matrix given in a graded form defined by (1.3), and perturbed to $\tilde{H} = D^*(A + \delta A)D$. If we write*

$$\widehat{A} = D^{-*}|H|D^{-1}, \tag{1.10}$$

*then (1.9) holds with*[1]

$$\eta_H = \|\delta A\| \, \|\widehat{A}^{-1}\|, \qquad \eta_K = 0. \tag{1.11}$$

**Proof.** Indeed,

$$\begin{aligned} |x^* \delta H x| &= |x^* D^* \delta A D x| = \|x^* D^* \delta A D x\| \leq \|x^* D^*\| \, \|\delta A\| \, \|Dx\| \\ &\leq \|\delta A\| \, \|\widehat{A}^{-1}\| \, x^* |H| x, \end{aligned} \tag{1.12}$$

as desired. ∎

[1] Although, the scaling with diagonal matrix has straightforward application in numerical analysis of various algorithms [6, 8, 14, 39, 53], a non-diagonal grading matrix may be of interest in cases of structured perturbations.

Another important class of perturbations is when $H$ is perturbed element-wise in the relative sense,

$$|\delta H_{ij}| \leq \varepsilon |H_{ij}|. \tag{1.13}$$

As we have already mentioned in Example 1.1, such perturbations occur when the matrix is stored into the computer with machine precision $\varepsilon$. Such perturbations are also caused by measurements since data are often determined to some relative accuracy.

**Corollary 1.2 (Veselić–Slapničar)** *Let $H$ and $K$ be as in Theorem 1.2, and let $\delta H$ be as in (1.13) and $\delta K = 0$. Then (1.9) holds with*

$$\eta_H = \varepsilon \| |A| \| \, \|\widehat{A}^{-1}\| \leq n\varepsilon \|\widehat{A}^{-1}\| \qquad \eta_K = 0. \tag{1.14}$$

**Proof.** By setting

$$D = \operatorname{diag}\left(\sqrt{|H|_{ii}}\right), \tag{1.15}$$

since $|\delta A_{ij}| \leq \varepsilon |A_{ij}|$, the relation (1.12) implies that (1.9) holds with

$$\eta_H = \varepsilon \| \, |A| \, \| \, \|\widehat{A}^{-1}\| \qquad \eta_K = 0.$$

Since $\widehat{A}_{ii} = 1$, we have $\|\widehat{A}^{-1}\| \leq \kappa(\widehat{A}) \leq n\|\widehat{A}^{-1}\|$, where $\kappa(A) \equiv \|A\| \, \|A^{-1}\|$ is the spectral condition number. Also, $\| \, |A| \, \| \leq n$ [39], which completes the proof. ■

The diagonal grading matrix $D$ from (1.15) is almost optimal in the sense that [43]

$$\kappa(\widehat{A}) \leq n \min_{\bar{D}} \kappa(\bar{D} |H| \bar{D}) \leq n\kappa(|H|) = n\kappa(H),$$

where the minimum is taken over all non-singular diagonal matrices.

Similarly, for more general perturbations we have the following corollary.

**Corollary 1.3 (Veselić–Slapničar)** *Let $H$ and $K$ be as in Theorem 1.2, and let $\delta H$ satisfy*

$$|\delta H_{ij}| \leq \varepsilon D_{ii} D_{jj}, \tag{1.16}$$

*and $\delta K = 0$. Then (1.9) holds with*

$$\eta_H = \varepsilon n \|\widehat{A}^{-1}\| \leq \varepsilon n \kappa(\widehat{A}), \qquad \eta_K = 0. \tag{1.17}$$

**Proof.** See [55, Theorem 2.17] ∎

The perturbation (1.16) is typically a consequence of rounding errors in various numerical algorithms, matrix factorizations, eigenvalue or singular value computations [39, 40, 1, 8].

Further, let us describe perturbation results for single Hermitian matrix $H$ given in the factorized form

$$H = GJG^*,$$

where $G$ has full column rank, and $J$ is Hermitian and non-singular.

As it is shown in [55, Theorem 3.3 pp. 105], the non vanishing eigenvalues of $H$ coincide with the eigenvalues of the pair $(G^*G, J^{-1})$.

**Corollary 1.4 (Veselić–Slapničar)** *Let $\lambda_i$ and $\tilde{\lambda}_i$ be the non-increasingly ordered eigenvalues of the matrices $H \equiv GJG^*$ and $\tilde{H} \equiv (I+E)^*GJG^*(I+E)$, respectively. If*

$$\eta_K \equiv 2\|E\| + \|E\|^2 < 1, \tag{1.18}$$

*then*

$$1 - \eta_K \leq \frac{\tilde{\lambda}_i}{\lambda_i} \leq 1 + \eta_K. \tag{1.19}$$

**Proof.** Non vanishing eigenvalues $\lambda_i$ are equal to reciprocal of non vanishing eigenvalues $\mu_i$ of the pairs $(J^{-1}, G^*G)$ and $(J^{-1}, G^*(I+E)(I+E)^*G)$. Let $\mu_i$ and $\tilde{\mu}_i$ be the non-increasingly ordered eigenvalues of the pairs $(J^{-1}, G^*G)$ and $(J^{-1}, G^*(I+E)(I+E)^*G)$. Then, from (1.9) follows

$$\frac{1-\eta_K}{1+\eta_H} \leq \frac{1/\tilde{\mu}_i}{1/\mu_i} \leq \frac{1+\eta_K}{1-\eta_H},$$

where $\eta_H$ bounds perturbation of matrix $J^{-1}$, and $\eta_K$ bounds perturbation of matrix $K \equiv G^*G$ into $K + \delta K \equiv G^*(I+E)(I+E)^*G$. Since $\eta_H = 0$, we only need to bound

$$\begin{aligned} |x^*\delta K x| &\leq |x^*G^*(E + E^* + EE^*)Gx| \leq \|x^*G^*\| \|E + E^* + EE^*\| \|Gx\| \\ &\leq (2\|E\| + \|E\|^2)\, x^*G^*Gx, \end{aligned}$$

which shows that $\eta_K = 2\|E\| + \|E\|^2$ as desired. ∎

If $H$ is positive definite, then $|H| = H$, and (1.14) and (1.17) reduce to the corresponding results from [8].

The following theorem contains a relative version of Hoffman-Wielandt-type of bounds for positive definite Hermitian matrix given in the graded form $H = D^*AD$ and perturbed to $H + \delta H = D^*(A + \delta A)D$.

**Theorem 1.3 (R. C. Li)** *Let $H = D^*AD$ and $H + \delta H = D^*(A + \delta A)D$ be Hermitian positive definite and let $\|A^{-1}\|\,\|\delta A\| < 1$ then*

$$\begin{aligned}\sqrt{\sum_{i=1}^{n}\frac{|\lambda_i-\tilde{\lambda}_i|^2}{\lambda_i\tilde{\lambda}_i}} &\leq \|A^{-1/2}\delta AA^{-1/2}\left(I + A^{-1/2}\delta AA^{-1/2}\right)^{-1/2}\|_F \\ &\leq \frac{\|A^{-1}\|\,\|\delta A\|_F}{\sqrt{1-\|A^{-1}\|\,\|\delta A\|}}. \end{aligned} \tag{1.20}$$

**Proof**. See [28, Theorem 3.2. pp. 962-963] or [27, Proposition 5.6 pp. 19]. ∎

Further, for the multiplicative perturbation $\tilde{H} = (I+E)^*H(I+E)$, we have the following result

**Theorem 1.4 (C. K. Li – Mathias)** *If $H$ and $H + \delta H = (I + E)^*H(I + E)$ are Hermitian and non-singular then*

$$\sqrt{\sum_{i=1}^{n}\frac{|\lambda_i-\tilde{\lambda}_i|^2}{|\lambda_i\tilde{\lambda}_i|}} \leq \|(I+E)-(I+E)^{-*}\|_F \tag{1.21}$$

**Proof**. See [27, Proposition 5.3 pp. 16]. ∎

In Chapter 4 we present relative linear and quadratic residual bound for the eigenvalues of Hermitian matrices.

Let $H \in \mathbf{C}^{n\times n}$ be a Hermitian matrix, $X \in \mathbf{C}^{n\times m}$ be an orthonormal matrix, and

$$M = X^*HX, \qquad R = HX - XM, \qquad \mathcal{X} = \mathcal{R}(X). \tag{1.22}$$

Furthermore, let

$$\lambda_1 \geq \ldots \geq \lambda_n \quad \text{ and } \quad \mu_1 \geq \ldots \geq \mu_m, \tag{1.23}$$

be the eigenvalues of $H$ and $M$, respectively.

The eigenvalues of $M$ are sometimes called Ritz values or Rayleigh-Ritz approximations of the eigenvalues of $H$. Ritz values are optimal in the sense that $\|R\|$ is minimized for $M = X^*HX$, that is, ([46, Theorem 1.15.IV] or [37, Theorem 11-4-5]) if we replace $M$ by another matrix $C$ we can only increase the norm of $R$,

$$\|R\| = \|HX - XM\| \leq \|HX - XC\|,$$

for all matrices $C$ of order $m$.

Moreover, one can always find $m$ eigenvalues of $H$ that are within absolute distance $\|R\|$ of the Ritz values ([37, Theorem 11-5-1])

$$\max_{1\le j\le m} |\lambda_{\tau(j)} - \mu_j| \le \|R\|,$$

for some permutation $\tau$. There is a similar residual bound given in the Frobenius norm ([46, Corollary 4.15.IV])

$$\sqrt{\sum_j (\lambda_{\tau(j)} - \mu_j)^2} \le \|R\|_F.$$

The above bounds measure absolute distance between eigenvalues, thus they belong to classical or absolute perturbation theory.

Drmač [12] has derived a relative residual error bound which is expressed in terms of angles between the relevant subspaces, $\mathcal{X} = \mathcal{R}(X)$, $\mathcal{Y} = \mathcal{R}(HX)$, $\mathcal{Z} = \mathcal{R}(H^{-1}X)$. The main results from [12] are described in the following theorem.

**Theorem 1.5 (Drmač)** *[12, Theorem 3, Porposition 5] There are at least $m$ eigenvalues $\lambda_{i_k}$, $k = 1, \ldots, m$ of $H$ for which*

$$\begin{aligned} \frac{|\lambda_{i_k} - \mu_k|}{|\lambda_{i_k}|} &\le \|(I - P_{\mathcal{X}})P_{\mathcal{Y},\mathcal{Z}} + P_{\mathcal{X}}(I - P_{\mathcal{Y},\mathcal{Z}})\| \\ &\le \sin\theta + \tan\phi, \quad k = 1, \ldots, m, \end{aligned} \tag{1.24}$$

*holds, where $P_{\mathcal{X}}$ denotes the orthogonal projector on $\mathcal{X}$, $P_{\mathcal{Y},\mathcal{Z}}$ is the projector on $\mathcal{Y}$ along $\mathcal{Z}^{\perp}$, $\theta$ is the maximal acute principal angle between $\mathcal{X}$ and $\mathcal{Y}$ and $\phi$ is the maximal acute principal angle between $\mathcal{Y}$ and $\mathcal{Z}$. We assume that the right hand side in (1.24) is less than one.*

**Theorem 1.6 (Drmač)** *[12, Theorem 6] Let $H = LL^*$ be positive definite. Let*

$$\mathcal{Y}_L = L^*\mathcal{X}, \qquad \mathcal{Z}_L = L^{-1}\mathcal{X},$$

*and let $\psi$ be the minimal acute principal angle between $\mathcal{Y}_L$ and $\mathcal{Z}_L$. There are at least $m$ eigenvalues $\lambda_{i_k}$, $k = 1, \ldots, m$ of $H$ for which*

$$\frac{|\lambda_{i_k} - \mu_k|}{|\lambda_{i_k}|} \le \frac{\sin\psi}{1 - \sin\psi} \quad k = 1, \ldots, m, \tag{1.25}$$

*provided that right hand side in (1.25) is less than one.*

More about existing linear residual bounds and quadratic residual bound is given in Chapter 4.

As we mentioned at the beginning of this introduction, we also consider the perturbations of eigenvectors, that is, invariant subspaces of non-singular indefinite Hermitian matrices under additive and multiplicative perturbations.

Let $\mathcal{U}$ and $\tilde{\mathcal{U}}$ be two invariant subspaces of the same dimension of Hermitian matrices $H$ and $\tilde{H}$, respectively. Let the columns of $U_\perp$ and $\tilde{U}$ form an orthonormal basis for $\mathcal{U}_\perp$ and $\tilde{\mathcal{U}}$. Let $Q\Sigma V^*$ be a singular value decomposition of the matrix $U_\perp^*\tilde{U}$. Since, the diagonal entries of the matrix $\sin\Theta(U,\tilde{U}) \equiv \Sigma$ are the sines of canonical angles between the subspaces $\mathcal{U}$ and $\tilde{\mathcal{U}}$ [46, Corollary I.5.4] we can express the distance between subspaces $\mathcal{U}$ and $\tilde{\mathcal{U}}$ in terms of $\|U_\perp^*\tilde{U}\|$.

There are two different approaches to derivation of the bounds for $\|U_\perp^*\tilde{U}\|$. The *classical approach* with absolute bounds and *relative approach* with relative bounds. The absolute bounds, roughly speaking, contain the ratio between $\|\tilde{H} - H\|$ and the absolute separation between eigenvalues, while the relative bounds contain the ratio between $\|\,|H|^{-1/2}(\tilde{H} - H)|H|^{-1/2}\|$ and the relative separation between the eigenvalues. Now we present some of the well-known results from both approaches. For more details see the review article [23].

Let $H$ be $n \times n$ non-singular indefinite Hermitian matrix, and let $\tilde{H} = H + \delta H$ be its perturbation. Let

$$H = U\Lambda U^* \;=\; \begin{bmatrix} U_1 & U_2 \end{bmatrix} \begin{bmatrix} \Lambda_1 & 0 \\ 0 & \Lambda_2 \end{bmatrix} \begin{bmatrix} U_1^* \\ U_2^* \end{bmatrix},$$

be its eigenvalue decomposition, where

$$\Lambda_1 = \operatorname{diag}(\lambda_{i_1}, \lambda_{i_2}, \cdots, \lambda_{i_k}), \quad \Lambda_2 = \operatorname{diag}(\lambda_{j_1}, \lambda_{j_2}, \cdots, \lambda_{j_l}).$$

If $\Lambda_1$ and $\Lambda_2$ have no common eigenvalues, than $U_1$ and $U_2$ both span simple invariant subspaces according to [46, Definition V.1.2.]. We assume the similar notation for perturbed quantities. Let

$$R = (H + \delta H)U_1 - U_1\Lambda_1$$

be corresponding residual. The following theorem belongs to the *classical approach.*

**Theorem 1.7 (Davis–Kahan)** *Let $H$ has the spectral decomposition (1.26) (let $\tilde{H}$ has similar spectral decomposition). If*

$$\delta = \min|\sigma(\Lambda_1) - \sigma(\tilde{\Lambda}_2)| > 0$$

*then*

$$\|\sin\Theta(U_1,\tilde U_1)\|_F \equiv \|\tilde U_2^* U_1\|_F \;\le\; \frac{\|R\|_F}{\delta} \le \frac{\|\delta H\|_F}{\delta}.$$

**Proof.** See [4, Section 2] or [46, Theorem 3.4. V.3.3]. ∎

Similar bound exists for any unitary invariant norm, but with a little stronger assumption on spectra $\sigma(\Lambda_1)$ and $\sigma(\Lambda_2)$. The following theorem contains such bound (but for the sake of simplicity we state the result in 2-norm).

**Theorem 1.8 (Davis–Kahan)** *Let $H$ and $\tilde H$ be as in Theorem 1.7. Suppose that*

$$\sigma(\Lambda_1) \subset [\alpha,\beta],$$

*and for some $\delta > 0$,*

$$\sigma(\Lambda_2) \subset \mathbf{R} \setminus [\alpha-\delta,\beta+\delta].$$

*Then*

$$\|\sin\Theta(U_1,\tilde U_1)\| \equiv \|\tilde U_2^* U_1\| \;\le\; \frac{\|R\|}{\delta} \le \frac{\|\delta H\|}{\delta}.$$

**Proof.** See [4, Section 2], [46, Theorem 3.6. V.3.3] or [24, Theorem 3.3]. ∎

The classical perturbation bounds are sometimes too pessimistic (see Example 1.2), especially for subspaces which belongs to close eigenvalues. For such case the relative perturbation bounds can be better.

Let $H = DAD$ be positive definite matrix with eigenvalue decomposition (1.26). Let $\tilde H = D^*(A+\delta A)D$ be positive definite with similar eigenvalue decomposition. In [34, Theorem 1] or [30, Theorem 3.3] we have the following theorem.

**Theorem 1.9 (R. C. Li)** *Let $H = D^*AD$ and $\tilde H = D^*(A+\delta A)D$ be positive definite Hermitian matrices, and let $\|A^{-1}\|\|\delta A\| < 1$. Then the canonical angles between the subspaces spanned by $U_1$ and $\tilde U_1$ are bounded by*

$$\|\sin\Theta\|_F \equiv \|\tilde U_2^* U_1\|_F \;\le\; \frac{\|A^{-1}\|\|\delta A\|_F}{\sqrt{1-\|A^{-1}\|\|\delta A\|}} \cdot \frac{1}{\displaystyle\min_{\substack{1\le p\le k\\ 1\le q\le l}} \frac{|\lambda_{i_p}-\tilde\lambda_{j_q}|}{\sqrt{|\lambda_{i_p}\tilde\lambda_{j_q}|}}}, \qquad (1.26)$$

*provided that the above minimum is greater than zero.*

**Proof**. See [30, Theorem 3.3] or [34, Theorem 1]. ∎

The following example shows the difference between absolute and relative bounds on a positive definite matrix.

**Example 1.2** *Let $H$ be diagonal matrix*

$$H = \begin{bmatrix} \lambda_1 & & \\ & 2\,\lambda_1 & \\ & & \lambda_2 \end{bmatrix},$$

*where $\lambda_2 >> \lambda_1 > 0$. Let $\tilde{H}$ be perturbed matrix produced by the storing of $H$ in floating point arithmetic,*

$$\tilde{H} = \begin{bmatrix} \lambda_1(1+\varepsilon_1) & & \\ & 2\lambda_1(1+\varepsilon_1) & \\ & & \lambda_2(1+\varepsilon_2) \end{bmatrix},$$

*where $|\varepsilon_i| \le \varepsilon$ and $\varepsilon > 0$ reflects the machine accuracy. Suppose that we wish to bound the sine of the largest canonical angle between invariant subspace spanned by $U = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}^T$ and its perturbation $\tilde{U}$ (actually $\tilde{U} = U$). From Theorem 1.8 it follows that the absolute bound for the largest canonical angle is given by*

$$\|\sin\Theta\| \le \frac{\|\tilde{H} - H\|}{\mathrm{absgap}},$$

*where* $\mathrm{absgap} = \min\{|\lambda_2(1+\varepsilon_2) - \lambda_1|, |2\lambda_1(1+\varepsilon_1) - \lambda_1|\}$. *If $\lambda_1 = 10^{-k}$ and $\lambda_2 = 10^k$, then absolute bound is*

$$\|\sin\Theta\|_F \le 10^{2k}\frac{\varepsilon_2}{1+2\varepsilon_1}.$$

*From Theorem 1.9 it follows that relative perturbation bound for $\|\sin\Theta\|_F$ in positive definite case is given by*

$$\|\sin\Theta\|_F \le \frac{\eta_F}{\sqrt{1-\eta_2}} \cdot \frac{1}{\mathrm{relgap}},$$

*where $\eta_2 = \|H^{-1/2}\delta H H^{-1/2}\|$ and $\eta_F = \|H^{-1/2}\delta H H^{-1/2}\|_F$ and*

$$\mathrm{relgap} = \min_{\lambda\in\Lambda_1, \tilde{\lambda}\in\tilde{\Lambda}_2} \frac{|\tilde{\lambda} - \lambda|}{\sqrt{\tilde{\lambda}\,\lambda}},$$

*where $\Lambda_1 = \lambda_1$ and $\Lambda_2 = \mathrm{diag}\,(2\,\tilde{\lambda}_1, \tilde{\lambda}_2)$. Thus, $\|\sin\Theta\|_F$ is bounded by*

$$\|\sin\Theta\|_F \le \frac{\varepsilon}{\sqrt{1-\varepsilon}} \cdot \frac{\sqrt{2(1+\varepsilon)}}{|1+2\,\varepsilon_2|}$$

The last example shows that relative bounds can sometimes be much sharper than absolute bounds. This is not a true in general (see [24] or [23]).

Relative perturbation bounds for eigenvectors of simple eigenvalues were given for scaled diagonally dominant matrices in [1], and for positive definite matrices in [8]. The bound for invariant subspace which corresponds to a single, possibly multiple, eigenvalue of an indefinite Hermitian matrix was given in [55, Theorem 2.48]. This bound, given in terms of projections which are defined by Dunford integral as in [26, Section II.1.4], is generalized to invariant subspaces which correspond to a set of neighboring eigenvalues in [50]. In [29, Theorems 4.5 and 4.6] two bounds for invariant subspaces of a graded positive semi-definite matrix were given. [23] gives an overview of existing classical and relative $\sin\theta$ theorems for invariant subspaces.

## 1.2 Relation between the existing and the new results

In this Section we present relations between the existing and the new results from this thesis.

First we describe a major difference between positive definite case which is considered in Theorem 1.3, [28, Theorem 3.2.], [27, Proposition 5.6 pp. 19] and Theorem 1.9, Theorem 1.6 or [12, Theorem 3] and the Hermitian indefinite case which we consider in Theorem 3.1 (see Table 1.1), Theorem 3.3 (see Table 1.5), Theorem 4.1 and Theorem 4.3 (see Table 1.2). Similar differences hold between the positive definite eigenvalue problem and the hyperbolic eigenvalue problem which is considered in Chapter 5. We show how the number $\|F\|^2$ can be used as a measure for the difference between positive definite and indefinite case, where $F$ is a $J$-unitary matrix, that is,

$$F^*JF = J, \qquad J = I_l \oplus (-I_{n-l}). \tag{1.27}$$

If $H$ is positive definite, then small perturbations of the type (1.13) and (1.16) cause small relative changes in eigenvalues if and only if $\kappa(\widehat{A}) \equiv \kappa(A)$ is small [8]. If $H$ is indefinite, then from (1.14) and (1.17) it follows that small $\kappa(\widehat{A})$ implies small relative changes in eigenvalues. However, these changes can be small even if $\kappa(\widehat{A})$ is large [55, 39]. Although such examples can be successfully analyzed by perturbations through factors as in [55], this shows that the indefinite case is essentially more difficult than the positive definite one.

Now we are going to derive a number which can be used as the measure of difference between positive definite and indefinite case.

Let $\tilde{H} = (I+E)^*GJG^*(I+E)$ be a multiplicative perturbation of an indefinite Hermitian matrix $H = GJG^*$ by multiplicative perturbation. If we substitute the matrix $J$ by any positive definite matrix, the bound (1.19) remains the same. This shows, that in the case of multiplicative perturbations, there is no difference between positive definite and indefinite case.

Unfortunately this is not a case with additive perturbations. We illustrate this difference for non-singular matrix $H$ given in the graded form $H = D^*AD$ which is perturbed to $\tilde{H} = D^*(A + \delta A)D$. As we already showed in Corollary 1.1 for eigenvalues of $H$ we have the following bound

$$1 - \eta \leq \frac{\tilde{\lambda}_i}{\lambda_i} \leq 1 + \eta, \tag{1.28}$$

where $\eta = \|\delta A\| \|\widehat{A}^{-1}\|$.

Let $A = Q|\Delta|^{1/2}J|\Delta|^{1/2}Q^*$ be the eigenvalue decomposition of $A$. Here $Q$ is unitary, $|\Delta|$ is diagonal with positive diagonal elements, and $J$ is diagonal with $\pm 1$ on its diagonal ($J$ is inertia of $A$, that is, inertia of $H$). Note, that we can write $H = GJG^*$, where $G = D^*Q|\Delta|^{1/2}$. Veselić and Slapničar showed [55, Theorem 3.3] existence of a non-singular matrix $F$ such that

$$F^*G^*GF = \Sigma^2, \qquad F^*JF = J.$$

Here $\Sigma$ is diagonal and the $i$-th eigenvalue of $H$ is given by $\lambda_i = (\Sigma_{ii})^2 J_{ii}$. The matrix $U = GF\Sigma^{-1}$ is unitary and its $i$-th column is $i$-th eigenvector of $H$. The matrices which satisfy the equality $F^*JF = J$ are called $J$-unitary.

Further, it is easy to show that $|H| = GFF^*G^*$. Now we can bound $\|\widehat{A}^{-1}\|$. Indeed,

$$\|\widehat{A}^{-1}\| = \|D|H|^{-1}D^*\| = \|DG^{-*}F^{-*}F^{-1}G^{-1}D^*\| \leq \|F^{-1}\|^2 \cdot \|A^{-1}\|.$$

As we show latter, for $J$-unitary $F$ it holds that $\|F^{-1}\| = \|F\|$. Thus, (1.28) holds with

$$\eta \leq \|F\|^2 \cdot \|A^{-1}\| \|\delta A\|. \tag{1.29}$$

If $H$ is positive definite, then $J = I$ and $F$ is unitary and the above result is equal to the corresponding results from [8].

Thus, we can use the number $\|F\|^2$ (note that $\kappa(F) = \|F\|^2$) as a measure of the difference between positive and indefinite case under additive perturbations.

Note that, if we consider perturbations of eigenvalues of Hermitian matrix $H = GJG^*$ which is perturbed to $\tilde{H} = (I + E)^*G\widehat{G}J\widehat{G}^*G(I + E)$, then from (1.19), (1.18) and (1.28), (1.29) it follows

$$\frac{\tilde{\lambda}_i}{\lambda_i} \leq 1 + \eta_M + \eta_A + \eta_M\eta_A, \tag{1.30}$$

where $\eta_M = 2\|E\| + \|E\|^2$, $\eta_A = \|F\|^2\|\widehat{G}J\widehat{G}^* - J\|$. Here $F$ is hyperbolic eigenvector matrix of the pair $(G^*G, J)$, that is

$$F^*G^*GF = |\Lambda| \qquad F^*JF = J, \tag{1.31}$$

where $|\Lambda| = \mathrm{diag}\,(|\lambda_1|, \ldots, |\lambda_n|)$.

In analogy with above considerations, in [9] perturbation of eigenvalues of Hermitian matrix $H = GJG^*$ perturbed to $H + \delta H = D_1GD_2JD_2^*G^*D_1^*$ has been considered. The following first order perturbation bound holds

$$\frac{1 - 2\|I - D_2\|_2\,\kappa_{rel}(\lambda_i)}{\|D_1\|^2} \leq \frac{\tilde{\lambda}_i}{\lambda_i} \leq \|D_1\|^2\,(1 + 2\|I - D_2\|_2\,\kappa_{rel}(\lambda_i))\,, \tag{1.32}$$

where $\kappa_{rel}(\lambda_i)$ is *relative condition number of* $\lambda_i$ defined by

$$\kappa_{rel}(\lambda_i) = \frac{|u_i^*GG^*u_i|}{|u_i^*GJG^*u_i|},$$

here $u_i$ denotes $i$-th eigenvector of the matrix $H$ corresponding to eigenvalue $\lambda_i$.

Note, for (1.31) follows that $U = GF|\Lambda|^{-1/2}$. Thus, let $G = U|\Lambda|^{1/2}V^*$ , where $U$ is unitary, $|\Lambda|$ is diagonal with absolute values of eigenvalues of $H$ and $V = \begin{bmatrix} v_1 & \ldots & v_n \end{bmatrix}$ is $J$-unitary, where $V^* = F$ (this is *hyperbolic singular value decomposition* (HSVD) introduced in [35] which is described in Section 2.3.2). Then from the identity $G^*u_i = \sqrt{|\lambda_i|}v_i$ it follows that

$$\kappa_{rel}(\lambda_i) = |v_i^*v_i|.$$

Note that if $J = I$, that is, in the positive definite case, in the bound (1.30) we have $\|F\| = 1$, and in the bound (1.32) we have $\kappa_{rel}(\lambda_i) = 1$. Further, if $\widehat{G} \equiv D_2 = I$, that is, in the case of multiplicative perturbations only, the bounds (1.30) and (1.32) do not depend on $\kappa(F)$ or $\kappa_{rel}(\lambda_i)$, respectively. Thus the same comment as one after the bound (1.29) holds for both bounds.

We pointed out a major difference between positive definite and indefinite case, thus, our first result presented in Chapter 2 Section 2.1 is about CS-decomposition of $J$-unitary matrix $F$ with partition

$$F = \begin{array}{c c} & \begin{array}{cc} l & n-l \end{array} \\ \begin{array}{c} l \\ n-l \end{array} & \begin{bmatrix} F_{11} & F_{12} \\ F_{21} & F_{22} \end{bmatrix} \end{array}.$$

If $2l \leq n$ then there exist unitary matrices $U = \text{diag}\,(U_{11}, U_{22})$ and $V = \text{diag}\,(V_{11}, V_{22})$, where $U_{11}$ and $V_{11}$ are $l \times l$ unitary, such that

$$\begin{bmatrix} U_{11}^* & 0 \\ 0 & U_{22}^* \end{bmatrix} \begin{bmatrix} F_{11} & F_{12} \\ F_{21} & F_{22} \end{bmatrix} \begin{bmatrix} V_{11} & 0 \\ 0 & V_{22} \end{bmatrix} = \begin{array}{c} \begin{array}{ccc} l & l & n-2l \end{array} \\ \begin{bmatrix} \Gamma & \Sigma & 0 \\ \Sigma & \Gamma & 0 \\ 0 & 0 & I \end{bmatrix} \end{array}, \tag{1.33}$$

where

$$\begin{aligned} \Gamma &= \text{diag}\,(\gamma_1, \ldots, \gamma_l) \geq 1 \\ \Sigma &= \text{diag}\,(\sigma_1, \ldots, \sigma_l) \geq 0 \\ & \Gamma^2 - \Sigma^2 = I. \end{aligned}$$

If $2l > n$ then

$$\begin{bmatrix} U_{11}^* & 0 \\ 0 & U_{22}^* \end{bmatrix} \begin{bmatrix} F_{11} & F_{12} \\ F_{21} & F_{22} \end{bmatrix} \begin{bmatrix} V_{11} & 0 \\ 0 & V_{22} \end{bmatrix} = \begin{array}{c} \begin{array}{ccc} l & n-2l & l \end{array} \\ \begin{bmatrix} \Gamma & 0 & \Sigma \\ 0 & I & 0 \\ \Sigma & 0 & \Gamma \end{bmatrix} \end{array}. \tag{1.34}$$

The above results are presented in Theorem 2.1. When $F$ is unitary then Theorem 2.1 reduces to [46, Theorem 5.1 I.5.1], with $\Gamma^2 + \Sigma^2 = I$. For $n = 2l$ Theorem 2.1 reduces to [45, Theorem 1].

Let $F$ be $J$-unitary matrix which simultaneously diagonalises the pair $(G^*G, J)$ where $G$ is a non-singular matrix, and let $\tilde{F}$ be $J$-unitary matrix which simultaneously diagonalises the pair $(\tilde{G}^*\tilde{G}, J)$ where $\tilde{G}$ is a non-singular matrix obtained from $G$ by perturbation. Using the result on CS-decomposition of $J$-unitary matrix, in Chapter 2 Section 2.4 we present perturbation bound for $\|\tilde{F}\|$ in the terms of $\|F\|$. We consider two kinds of perturbations of matrix $G$:

1. $\tilde{G} = G\,(I + E_R)$,
2. $\tilde{G} = (I + E_L)\,G$.

Our first bound is given in Lemma 2.1. Let $G$ and $\tilde{G} = G\,(I + E_R)$ be two $n \times n$ non-singular matrices. Let $F$ and $\tilde{F}$ be $J$-unitary matrices which simultaneously diagonalise the pairs $(G^*G, J)$ and $(\tilde{G}^*\tilde{G}, J)$, respectively. Define $\gamma = \|E_R\|_F/(1 - \|E_R\|)$. If

$$\gamma \|F\|^2 \; < \; \frac{1}{4},$$

then

$$\|\tilde{F}\| \leq \frac{\|F\|}{\sqrt{1-4\gamma\|F\|^2}}.$$

The second bound is given in Lemma 2.2. Let $G$ and $\tilde{G} = (I + E_L)G$ be two $n \times n$ non-singular matrices. Let $F$ and $\tilde{F}$ be $J$-unitary matrices which simultaneously diagonalise the pairs $(G^*G, J)$ and $(\tilde{G}^*\tilde{G}, J)$, respectively. Define $\eta = \|E_L\|_F/(1 - \|E_L\|)$. If $\eta < 1$, then

$$\|\tilde{F}\| \leq \left(\eta + \sqrt{1+\eta^2}\right) \|F\|.$$

Further, in Chapter 3 Section 3.1.1 we present a generalization of Theorem 1.3 on indefinite non-singular Hermitian matrices given in a graded form $H = D^*AD$ and perturbed to $H + \delta H = D^*(A + \delta A)D$.

In Theorem 3.1 Section 3.1.1 we present the following result (see Table 1.1):

Let $H = D^*AD$ and $H + \delta H = D^*(A + \delta A)D$ be Hermitian non-singular matrices. If $\|A^{-1}\|\|\delta A\| < 1$ then

$$\sqrt{\sum_{i=1}^{n} \frac{|\lambda_i - \tilde{\lambda}_i|^2}{|\lambda_i\tilde{\lambda}_i|}} \leq \|V\|\|\tilde{V}\| \frac{\|A^{-1}\|\,\|\delta A\|_F}{\sqrt{1-\|A^{-1}\|\,\|\delta A\|}},$$

provided that the right hand side is less than 2. Here $V$ and $\tilde{V}$ are hyperbolic eigenvector matrices of the pairs $(G^*G, J)$ and $(\tilde{G}^*\tilde{G}, J)$, respectively.

As one can see the upper result is a proper generalization of (1.20) on indefinite non-singular matrices since for positive definite case $V$ and $\tilde{V}$ are unitary.

Further, in Chapter 3 Section 3 we present perturbation bound for the multiplicative perturbation $\tilde{H} = (I + E)^*H(I + E)$, where $H$ is non-singular Hermitian matrix. We use similar technique as in the case of additive perturbations. Our result is presented in Theorem 3.2 (see Table 1.1).

In Chapter 3.2 we present a generalization of the result of Theorem 1.9 [30, Theorem 3.3] or [34, Theorem 1] or [30, Theorem 3.3] and give the bound for $\|\sin\Theta\|_F$ for a graded non-singular indefinite matrix. Our results are, therefore, relative variants of the well-known $\sin\Theta$ theorems, that is, Theorems 1.7 and 1.8 [4, Section 2], [46, Theorem V.3.4]. Our results also generalize the ones from [55, 50] to subspaces which correspond to arbitrary set of eigenvalues. It is given in Theorem 3.3 Chapter 3.2 Section 3.2.1 (see Table 1.5).

Note that this result generalized corresponding result from Theorem 1.9 [30, Theorem 3.3] to indefinite Hermitian matrices. Indeed, for $A$ positive definite $V_1$ and $\tilde{V}_2$ have orthonormal columns, and the bound (1.26) is equal to the upper bound (3.10) from Theorem 1.9 [30, Theorem 3.3].

Further, in Chapter 4 we present several linear and quadratic residual bounds for Ritz values of indefinite Hermitian matrix (see Table 1.2).

The bound from Theorem 4.1 from Chapter 4 Section 4.1 is a proper generalization of the bound (14) of 1.6 [12, Theorem 6] on indefinite Hermitian matrices. Our quadratic residual bound is generalization of quadratic bound from [13, Theorem 2.1] to indefinite possible singular Hermitian matrices.

In Chapter 5 we consider the hyperbolic eigenvalue problem

$$Hx_i = \lambda_i J x_i, \qquad i = 1, 2, \cdots, n,$$

where $H$ is a $n \times n$ Hermitian positive-semidefinite matrix, and $J$ is non-singular matrix of the form $J = \operatorname{diag}(\pm 1)$. We derive the relative version of Hoffman-Wielandt-type of bound for eigenvalues of the pair $(H, J)$ under two kinds of perturbations, additive perturbation, where $H = D^*AD$ is perturbed to $H + \delta H = D^*(A + \delta A)D$, and multiplicative perturbation, where $H$ is perturbed to $H + \delta H = (I + E)^* H (I + E)$. We also give the relative versions of $\sin \Theta$ theorem for invariant subspaces spanned by columns of $X_1$ and $\tilde{X}_1$, where $X$ is non-singular matrix such that $X = \begin{bmatrix} X_1 & X_0 \end{bmatrix}$ such that

$$\begin{aligned} \begin{bmatrix} X_1^* \\ X_0^* \end{bmatrix} H \begin{bmatrix} X_1 & X_0 \end{bmatrix} &= \begin{bmatrix} |\Lambda_r| & \\ & 0_{n-r} \end{bmatrix} \equiv |\Lambda|, \\ \begin{bmatrix} X_1^* \\ X_0^* \end{bmatrix} J \begin{bmatrix} X_1 & X_0 \end{bmatrix} &= \begin{bmatrix} \widehat{J}_1 & \\ & \widehat{J}_2 \end{bmatrix} \equiv \widehat{J}, \end{aligned}$$

where $|\Lambda_r| = \operatorname{diag}(|\lambda_1|, \ldots, |\lambda_r|)$, and $\widehat{J}$ is some permutation of $J$. $\tilde{X}$ is perturbed $X$ with similar structure (see Table 1.3 and Table 1.6).

In Chapter 6 we consider the generalized eigenvalue problem of the form

$$Hx = \lambda M x,$$

where $H, M \in \mathbf{C}^{n \times n}$ Hermitian and $x \in \mathbf{C}^n$, $x \neq 0$.

We present a couple of classical and relative perturbation bounds for eigenvalues and eigenspaces of the pair $(H, M)$ under the additive perturbation $(H + \delta H, M + \delta M)$.

Our bounds are given for the matrix pairs $(H, M)$ for which exists a non-singular matrix $X$ such that

$$X^* H X = \Omega, \quad X^* M X = J,$$

where

$$\begin{aligned} \Omega = \operatorname{diag}(\omega_1, \ldots, \omega_n) &\qquad J = \operatorname{diag}(j_1, \ldots, j_n), \\ \omega_i \in \mathbf{R} &\qquad j_i \in \{-1, 1\}, \end{aligned}$$

for $i = 1, \ldots, n$ (see Table 1.4 and Table 1.7).

# Chapter 2

# Indefinite CS-decomposition and its applications

As we have seen in the Section 1.2 there is a difference between positive definite and indefinite case. It comes out that we can use the number $\|F\|^2$ as a measure for the difference between positive definite and indefinite case, where $F$ is $J$-unitary. Thus, it is of our interest to study more about $J$-unitary matrices and their properties.

The first section of this chapter is devoted to the $J$-unitary matrices. In this section we present generalization of CS-decomposition of unitary matrices on the $J$-unitary matrices, which we call *indefinite CS-decomposition*. Then using this result, in the next section, we compare two subspaces, of the same dimension, spanned by columns of a $J$-unitary matrix and we define a *hyperbolic canonical angle* between two subspaces. Further, we show how one can obtain some of the well known properties of $J$-unitary matrices using the *indefinite CS-decomposition* such as computing the norm of a $J$-unitary matrix or deriving a bound for the maximal *hyperbolic canonical angle* between two subspaces. Finally, in Section 2.4 we present two perturbation bounds for $\|\tilde{F}\|$ in terms of $\|F\|$.

## 2.1 CS decomposition of a $J$-unitary matrix

Before we state the main result of this section we will present some of the well known properties of $J$-unitary matrices for diagonal $J$ with $\pm 1$ on its diagonal.

As we already mentioned, the non-singular matrix $F$ with property

$$F^* J F = J$$

is called $J$-unitary. Such a matrix has the following properties:

1. $\|F\| = \|F^{-1}\|$. Moreover, the singular values of $F$ come in pairs of reciprocals, $\{\sigma, 1/\sigma\}$. Also, note that $|\det(F)| = 1$.

2. If $J$ is given as $J = I_l \oplus (-I_{n-l})$ then $J$-unitary $F$ has the following form:

$$F = \begin{bmatrix} (I + X^*X)^{1/2} & X^* \\ X & (I + XX^*)^{1/2} \end{bmatrix} \begin{bmatrix} U_+ & 0 \\ 0 & U_- \end{bmatrix}, \tag{2.1}$$

   where $X$ is rectangular matrix whose dimension corresponds to blocks of $J$, and $U_+$, $U_-$ are some orthogonal matrices of corresponding order. This property is from [53, Theorem A3.].

3. Let $F$ be partitioned accordingly,

$$F = \begin{bmatrix} F_{11} & F_{12} \\ F_{21} & F_{22} \end{bmatrix}.$$

   Then a straightforward computation shows that form (2.1) follows $\|F_{11}\| = \|F_{22}\|$, $\|F_{12}\| = \|F_{21}\|$, and

$$\|F\| = \|F_{21}\| + \|F_{11}\| = \|F_{21}\| + \sqrt{1 + \|F_{21}\|^2}.$$

The following theorem gives the CS-decomposition of $J$-unitary matrix $F$. We will show that all above properties of $J$-unitary matrices follows from this theorem. Moreover, we will present some other result which follows from CS-decomposition of $J$-unitary matrices, for instance, we will describe hyperbolic canonical angles and estimation of the norm of $J$-unitary matrix.

Without loss of generality we may assume that $J$ has the following form:

$$J \;=\; I_l \oplus (-I_{n-l}). \tag{2.2}$$

The result about the CS-decomposition of $J$-unitary matrix can be easily obtained from 2.1, but we present a slightly complicated version of the same result in the following theorem because it is more suitable for definition of a *hyperbolic canonical angle* between two subspaces.

**Theorem 2.1** *Let $F$ be $J$-unitary matrix, with the partition*

$$F = \begin{array}{c c} & \begin{array}{cc} l & n-l \end{array} \\ \begin{array}{c} l \\ n-l \end{array} & \begin{bmatrix} F_{11} & F_{12} \\ F_{21} & F_{22} \end{bmatrix} \end{array}. \tag{2.3}$$

*If $2l \leq n$, then there exist unitary matrices $U = \operatorname{diag}(U_{11}, U_{22})$ and $V = \operatorname{diag}(V_{11}, V_{22})$, where $U_{11}$ and $V_{11}$ are $l \times l$ unitary, such that*

$$\begin{bmatrix} U_{11}^* & 0 \\ 0 & U_{22}^* \end{bmatrix} \begin{bmatrix} F_{11} & F_{12} \\ F_{21} & F_{22} \end{bmatrix} \begin{bmatrix} V_{11} & 0 \\ 0 & V_{22} \end{bmatrix} = \begin{array}{c} \begin{array}{ccc} l & l & n-2l \end{array} \\ \begin{bmatrix} \Gamma & \Sigma & 0 \\ \Sigma & \Gamma & 0 \\ 0 & 0 & I \end{bmatrix} \end{array}, \tag{2.4}$$

*where*

$$\begin{aligned} \Gamma &= \operatorname{diag}(\gamma_1, \ldots, \gamma_l) \geq 1 \\ \Sigma &= \operatorname{diag}(\sigma_1, \ldots, \sigma_l) \geq 0 \\ & \Gamma^2 - \Sigma^2 = I. \end{aligned}$$

*If $2l > n$ then*

$$\begin{bmatrix} U_{11}^* & 0 \\ 0 & U_{22}^* \end{bmatrix} \begin{bmatrix} F_{11} & F_{12} \\ F_{21} & F_{22} \end{bmatrix} \begin{bmatrix} V_{11} & 0 \\ 0 & V_{22} \end{bmatrix} = \begin{array}{c} \begin{array}{ccc} l & n-2l & l \end{array} \\ \begin{bmatrix} \Gamma & 0 & \Sigma \\ 0 & I & 0 \\ \Sigma & 0 & \Gamma \end{bmatrix} \end{array}. \tag{2.5}$$

**Proof**. The proof is similar to the proof of [46, Theorem 5.1]. Let $U_{11}^* F_{11} V_{11} = \Gamma$ be the singular value decomposition of $F_{11}$, and suppose that

$$\Gamma = \operatorname{diag}(\Gamma_1, I_{l-k}),$$

where the diagonal elements of $\Gamma_1$ satisfy $\gamma_i > 1$, $i = 1, \ldots, k$ (since $F_{11}^* F_{11} = I + F_{21}^* F_{21}$). From the equality

$$V_{11}^* \begin{bmatrix} F_{11}^* & F_{21}^* \end{bmatrix} \begin{bmatrix} I_l & 0 \\ 0 & -I_{n-l} \end{bmatrix} \begin{bmatrix} F_{11} \\ F_{21} \end{bmatrix} V_{11} = I_l,$$

(where we have used the fact that $F_{11}^* F_{11} - F_{21}^* F_{21} = I_l$) we have

$$\Gamma^2 - V_{11}^* F_{21}^* F_{21} V_{11} = I_l,$$

or

$$V_{11}^* F_{21}^* F_{21} V_{11} = \operatorname{diag}(I + \Gamma_1^2, 0_{l-k}).$$

This means that the first $k$ columns of $F_{21}V_{11}$ are orthogonal and the last $l-k$ columns being zeros. Thus, there is a unitary matrix $\widehat{U}_{22}$ with dimension $(n-l)\times(n-l)$ such that

$$\widehat{U}_{22}^* F_{21} V_{11} = \begin{bmatrix} \Sigma \\ 0 \end{bmatrix},$$

where

$$\Sigma = \operatorname{diag}(\Sigma_1, 0_{l-k}), \tag{2.6}$$

and $\Sigma_1 = \operatorname{diag}(\sigma_1, ..., \sigma_k)$ has positive diagonal elements. Now we can write

$$\begin{bmatrix} U_{11}^* & 0 \\ 0 & \widehat{U}_{22}^* \end{bmatrix} \begin{bmatrix} F_{11} \\ F_{21} \end{bmatrix} V_{11} = \begin{bmatrix} \Gamma \\ \Sigma \\ 0 \end{bmatrix}.$$

Since

$$V_{11}^* \begin{bmatrix} F_{11}^* & F_{21}^* \end{bmatrix} \begin{bmatrix} U_{11} & 0 \\ 0 & \widehat{U}_{22} \end{bmatrix} \begin{bmatrix} I_l & 0 & \\ 0 & -I_{n-l} & 0 \end{bmatrix} \begin{bmatrix} U_{11}^* & 0 \\ 0 & \widehat{U}_{22}^* \end{bmatrix} \begin{bmatrix} F_{11} \\ F_{21} \end{bmatrix} V_{11} = I_l,$$

we must have

$$\gamma_i^2 - \sigma_i^2 = 1, \qquad i = 1, \ldots l. \tag{2.7}$$

In a similar manner we may determine a unitary matrix $V_{22}$ of a dimension $(n-l)\times(n-l)$ such that

$$U_{11}^* F_{12} V_{22} = \begin{bmatrix} T & 0 \end{bmatrix},$$

where $T = \operatorname{diag}(\tau_1, \ldots, \tau_l)$. Since

$$U_{11}^* \begin{bmatrix} F_{11} & F_{12} \end{bmatrix} \begin{bmatrix} V_{11} & 0 \\ 0 & V_{22} \end{bmatrix} \begin{bmatrix} I_l & 0 \\ 0 & -I_{n-l} \end{bmatrix} \begin{bmatrix} V_{11}^* & 0 \\ 0 & V_{22}^* \end{bmatrix} \begin{bmatrix} F_{11}^* \\ F_{21}^* \end{bmatrix} U_{11} = I_l,$$

we must have

$$\gamma_i^2 - \tau_i^2 = 1, \qquad i = 1, \ldots l$$

and it follows from (2.6) and (2.7) that $T = \Sigma$.

Set $\widehat{U} = \operatorname{diag}(U_{11}, \widehat{U}_{22})$ and $V = \operatorname{diag}(V_{11}, V_{22})$. Then the matrix $X = \widehat{U}^* F V$ has partition

$$X = \begin{array}{c} \\ k \\ l-k \\ k \\ l-k \\ n-2l \end{array} \begin{array}{c} \begin{array}{ccccc} k & l-k & k & l-k & n-2l \end{array} \\ \left[\begin{array}{ccccc} \Gamma_1 & 0 & \Sigma_1 & 0 & 0 \\ 0 & I & 0 & 0 & 0 \\ \Sigma_1 & 0 & X_{33} & X_{34} & X_{35} \\ 0 & 0 & X_{43} & X_{44} & X_{45} \\ 0 & 0 & X_{53} & X_{54} & X_{55} \end{array}\right] \end{array}.$$

Matrix $X$ is $J$-unitary ($X^*JX = J$), and $\Sigma_1$ has positive elements, thus

$$0 = (X^*JX)_{13} = \Gamma_1\Sigma_1 - \Sigma_1 X_{33} \quad \Rightarrow \quad X_{33} = \Gamma_1.$$

Further, note that

$$\begin{aligned} -I_k = (X^*JX)_{33} = {\Sigma_1}^2 - X_{33}^2 - X_{43}^* X_{43} - X_{53}^* X_{53} \\ \Rightarrow \quad X_{43}^* X_{43} + X_{53}^* X_{53} = 0. \end{aligned}$$

Since $\operatorname{tr}(X_{43}^* X_{43}) + \operatorname{tr}(X_{53}^* X_{53}) = 0$, we can conclude that $X_{43} = 0$ and $X_{53} = 0$. Similarly, we can show that $X_{34} = 0$ and $X_{35} = 0$. Therefore we can write matrix $X$ in the following form

$$X = \left[\begin{array}{ccccc} \Gamma_1 & 0 & \Sigma_1 & 0 & 0 \\ 0 & I & 0 & 0 & 0 \\ \Sigma_1 & 0 & \Gamma_1 & 0 & 0 \\ 0 & 0 & 0 & X_{44} & X_{45} \\ 0 & 0 & 0 & X_{54} & X_{55} \end{array}\right],$$

where

$$\left[\begin{array}{cc} X_{44} & X_{45} \\ X_{54} & X_{55} \end{array}\right]$$

is unitary matrix. Set

$$U_{33} = \left[\begin{array}{cc} X_{44} & X_{45} \\ X_{54} & X_{55} \end{array}\right] \in \mathbf{C}^{(n-l-k)\times(n-l-k)}.$$

Then we have

$$\operatorname{diag}(I_{l+k},\ U_{33}^*) X = \left[\begin{array}{ccccc} \Gamma_1 & 0 & \Sigma_1 & 0 & 0 \\ 0 & I & 0 & 0 & 0 \\ \Sigma_1 & 0 & \Gamma_1 & 0 & 0 \\ 0 & 0 & 0 & I & 0 \\ 0 & 0 & 0 & 0 & I \end{array}\right],$$

which is equivalent to (2.4).
Observe that

$$\mathrm{diag}\,(I_{l+k},\ \ U_{33}^*)X = \mathrm{diag}\,(I_{l+k},\ \ U_{33}^*)\widehat{U}^*FV.$$

Hence, if we set

$$\begin{aligned} U &= \widehat{U}\mathrm{diag}\,(I_{l+k},\ \ U_{33}) \\ &= \mathrm{diag}\,(U_{11},\widehat{U}_{22})\ \ \mathrm{diag}\,(I_l,\ \ \mathrm{diag}\,(I_k,U_{33})) \\ &= \mathrm{diag}\,(U_{11},\widehat{U}_{22}\,\mathrm{diag}\,(I_k,\ \ U_{33})) \\ &= \mathrm{diag}\,(U_{11},\ \ U_{22}), \end{aligned}$$

then $U^*FV$ has the form (2.4), where $U$ and $V$ are diagonal unitary block matrices. ■

In the following sections we describe some applications of $J$-unitary matrices.

## 2.2 Pairs of subspaces

Similarly as in the unitary case [46, Section I.5.3], we can describe relation between two subspaces of the same dimension in the $J$-unitary basis, that is, we can describe how two subspaces with $J$-unitary basis are situated with respect to one other.

Let $X$ and $Y$ be two $J$-unitary matrices, that is $X^*JX = J$ and $Y^*JY = J$. Further, let both be decomposed as

$$\begin{aligned} X = [\,X_+ \quad X_-\,] \ \text{ such that } \ X_+^*JX_+ = I_l, \quad X_-^*JX_- = -I_{n-l}, \\ Y = [\,Y_+ \quad Y_-\,] \ \text{ such that } \ Y_+^*JY_+ = I_l, \quad Y_-^*JY_- = -I_{n-l}. \end{aligned} \tag{2.8}$$

Then we have the following theorem.

**Theorem 2.2** *Let $X_+, Y_+ \in \mathbf{C}^{n\times l}$ be as in (2.8). If $2l \le n$, then there exist $J$-unitary matrix $Q$ and unitary matrices $U_{11}$ and $V_{11}$ such that*

$$QX_+U_{11} = \begin{array}{c c} & l \\ \begin{array}{c} l \\ l \\ n-2l \end{array} & \left[\begin{array}{c} I \\ 0 \\ 0 \end{array}\right] \end{array}, \tag{2.9}$$

$$QY_+V_{11} = \begin{array}{c c} & l \\ \begin{array}{c} l \\ l \\ n-2l \end{array} & \left[\begin{array}{c} \Gamma \\ \Sigma \\ 0 \end{array}\right] \end{array}, \tag{2.10}$$

*where*

$$\Gamma = \operatorname{diag}(\gamma_1, ..., \gamma_l) \qquad \text{and} \qquad \Sigma = \operatorname{diag}(\sigma_1, ..., \sigma_l)$$

*satisfy*

$$\begin{aligned} \gamma_1 \geq ... \geq \gamma_l \geq 1, \\ \sigma_1 \geq ... \geq \sigma_l \geq 0, \\ \gamma_i^2 - \sigma_i^2 = 1, \qquad i = 1, \ldots, l. \end{aligned} \tag{2.11}$$

*On the other hand, if* $2l > n$, *then* $Q$, $U_{11}$ *and* $V_{11}$ *can be chosen such that*

$$QX_+U_{11} = \begin{array}{c} \\ n-l \\ 2l-n \\ n-l \end{array} \begin{array}{c} \begin{array}{cc} n-l & 2l-n \end{array} \\ \begin{bmatrix} I & 0 \\ 0 & I \\ 0 & 0 \end{bmatrix} \end{array}, \tag{2.12}$$

$$QY_+V_{11} = \begin{array}{c} \\ n-l \\ 2l-n \\ n-l \end{array} \begin{array}{c} \begin{array}{cc} n-l & 2l-n \end{array} \\ \begin{bmatrix} \Gamma & 0 \\ 0 & I \\ \Sigma & 0 \end{bmatrix} \end{array}, \tag{2.13}$$

*where*

$$\Gamma = \operatorname{diag}(\gamma_1, ..., \gamma_{n-l}) \qquad \Sigma = \operatorname{diag}(\sigma_1, ..., \sigma_{n-l})$$

*satisfy*

$$\begin{aligned} \gamma_1 \geq ... \geq \gamma_{n-l} \geq 1, \\ \sigma_1 \geq ... \geq \sigma_{n-l} \geq 0, \\ \gamma_i^2 - \sigma_i^2 = 1, \qquad i = 1, ..., n-l. \end{aligned} \tag{2.14}$$

**Proof**. We will prove only the case $2l \leq n$. The case $2l > n$ can be proved similarly.

By Theorem 2.1, for $J$-unitary matrix

$$F = X^*JY = \begin{bmatrix} X_+^*JY_+ & X_+^*JY_- \\ X_-^*JY_+ & X_-^*JY_- \end{bmatrix}$$

there are unitary matrices $U_{11}, U_{22}, V_{11}$ and $V_{22}$ such that (2.4) holds. Here $X$ and $Y$ are partitioned as in (2.8). If we set

$$\widehat{X_1} = X_+U_{11}, \ \ \widehat{X_2} = X_-U_{22}, \qquad \text{and} \qquad \widehat{Y_1} = Y_+V_{11}, \ \ \widehat{Y_2} = Y_-V_{22},$$

and

$$\widehat{X} = \begin{bmatrix} \widehat{X_1} & \widehat{X_2} \end{bmatrix}, \quad \widehat{Y} = \begin{bmatrix} \widehat{Y_1} & \widehat{Y_2} \end{bmatrix},$$

then

$$\widehat{X}^* J \widehat{Y} = \begin{array}{c} \\ l \\ l \\ n-2l \end{array} \begin{array}{c} \begin{array}{ccc} l & l & n-2l \end{array} \\ \begin{bmatrix} \Gamma & \Sigma & 0 \\ \Sigma & \Gamma & 0 \\ 0 & 0 & I \end{bmatrix} \end{array}.$$

We can permute columns of $U_{ii}$ and $V_{ii}$ so that (2.11) holds. By setting $Q = \widehat{X}^* J$ we have proved equations (2.9) and (2.10), which completes the proof. ■

Theorem 2.2 has the same geometrical interpretation as Theorem I.5.2. from Stewart-Sun [46] in unitary case. If $\mathcal{X}_+$ and $\mathcal{Y}_+$ are $l$-dimensional subspaces of $\mathbf{C}^n$ spanned by the columns of $J$-unitary matrices $X_+$ and $Y_+$, respectively, that is, $X_+^* J X_+ = Y_+^* J Y_+ = I_l$, then we can transform $\mathbf{C}^n$ by $J$-unitary matrix $Q$ so that columns of the matrices

$$\begin{bmatrix} I \\ 0 \\ 0 \end{bmatrix} \qquad \text{and} \qquad \begin{bmatrix} \Gamma \\ \Sigma \\ 0 \end{bmatrix}$$

form the basis for $Q\mathcal{X}_+$ and $Q\mathcal{Y}_+$. The subspace spanned by columns of for which $\gamma_i = 0$, is the subspace of $Q\mathcal{Y}_+$ that is orthogonal to $Q\mathcal{X}_+$.

Let $X = \begin{bmatrix} X_+ & X_- \end{bmatrix}$ be $J$-unitary matrix as in (2.8). Let $\mathcal{X}_+$ be the subspace of $\mathbf{C}^n$ spanned by the columns of $X_+$, that is, $\mathcal{X}_+ = \mathcal{R}(X_+)$. Then $\mathcal{X}_+$ is called *positive subspaces corresponding to matrix $J$*. When matrix $J$ is fixed we will call $\mathcal{X}_+$ *positive subspace*. Similarly, the $n-l$-dimensional subspace $\mathcal{X}_-$ of $\mathbf{C}^n$ spanned by columns of $J$-unitary matrix $X_-$ ( that is, $\mathcal{X}_- = \mathcal{R}(X_-)$ where $X_-^* J X_- = -I_l$) we call negative subspaces corresponding to matrix $J$.

**Definition 2.1** *Let $\mathcal{X}_+$, and $\mathcal{Y}_+$ be two positive subspaces of $\mathbf{C}^n$ corresponding to matrix $J$, spanned by columns of $J$-unitary matrices $X_+$ and $Y_+$, respectively. Diagonal elements of matrix*

$$\Theta_h(\mathcal{X}_+, \ \mathcal{Y}_+) := \sinh^{-1} \Sigma,$$

*are* hyperbolic canonical angles *between $\mathcal{X}_+$ and $\mathcal{Y}_+$, where $\Sigma$ is defined as in Theorem 2.2.*

The following corollary shows how to compute *hyperbolic canonical angles* between two positive (or negative) subspace of the same dimension.

**Corollary 2.1** *Let $\mathcal{X}_+$ and $\mathcal{Y}_+$ be two positive subspaces corresponding to matrix $J$. Let columns of $X_-$ form a basis for the negative subspace $\mathcal{X}_-$ and let columns of $Y_+$ form basis for the positive subspace $\mathcal{Y}_+$. Then the nonzero singular values of $X_-^* J Y_+$ are nonzero hyperbolic sines of hyperbolic canonical angles between $\mathcal{X}_+$ and $\mathcal{Y}_+$.*

**Proof**. Let $X = \begin{bmatrix} X_+ & X_- \end{bmatrix}$ and $Y = \begin{bmatrix} Y_+ & Y_- \end{bmatrix}$ be two $J$-unitary matrices, with $\mathcal{R}(X_+) = \mathcal{X}_+$ and $\mathcal{R}(Y_+) = \mathcal{Y}_+$. If we write

$$W = X^* J Y = \begin{bmatrix} X_+^* J Y_+ & X_+^* J Y_- \\ X_-^* J Y_+ & X_-^* J Y_- \end{bmatrix},$$

then Theorem 2.1 implies that there exist unitary matrices $U_{11}, U_{22}, V_{11}$ i $V_{22}$ such that (2.4) holds. This gives

$$X_-^* J Y_+ = U_{22} \begin{bmatrix} \Sigma \\ 0 \end{bmatrix} V_{11}^*.$$

The last equality means that the non-zero singular values of the matrix $X_-^* J Y_+$ are equal to nonzero diagonal elements of the matrix $\Sigma$. ■

## 2.3 Some of applications of the CS-decomposition

In this section we will present a couple applications of Theorem 2.1 and Corollary 2.1. We will assume, without a loss of generality, that matrix $J$ has form (2.2) with $2l \leq n$, in all cases.

First, we derive a norm of $J$-unitary matrix $F$. Further, if $F$ is $J$-unitary matrix defined by the hyperbolic singular value decomposition of a non-singular matrix $G = U \Sigma F^*$ (in the next subsection we will shortly describe the hyperbolic singular value decomposition (HSVD), introduced in [35]), then show how to compute $\|F\|$ knowing only positive (or negative) eigenvalues of a matrix $GJG^*$ and the corresponding eigenvectors.

Finally, we present a bound for $\|F\|$ in the terms of the maximal *hyperbolic canonical angle* between two positive subspaces $\mathcal{R}(F_+)$ and $\mathcal{R}(V_+)$, where $F$ and $V$ are $J$-unitary. We also compute the bound for $\|F_2^* J V_1\|$, where $F_2 \neq F_-$ and $V_1 \neq V_+$.

First, we will show a simple formulae for deriving the singular values of $J$-unitary matrix $F$.

### 2.3.1 Singular values of $J$-unitary matrix

In this section we present a simple corollary of Theorem 2.1 which describe singular values of the $J$-unitary matrix $F$.

**Corollary 2.2** *Let $F$ be $J$-unitary matrix. Then for its singular values $\tau_i$ holds:*

$$\begin{aligned} \tau_i &= \gamma_i + \sigma_i, \quad i = 1, \dots, l \\ \tau_{l+i} &= \gamma_i - \sigma_i, \quad i = 1, \dots, l \\ \tau_k &= 1, \qquad\quad i = 2l+1, \dots, n. \end{aligned} \tag{2.15}$$

*In (2.15) $\gamma_i$ and $\sigma_i$ are defined as in Theorem 2.1, that is, $\gamma_i^2 - \sigma_i^2 = 1$ for $i = 1, \dots, l$.*

**Proof**. The singular values of the matrix $F$ are square roots of the eigenvalues of a matrix $F^*F$. From (2.4) follows that

$$F^*F = \begin{bmatrix} V_{11}^* & 0 \\ 0 & V_{22}^* \end{bmatrix} \begin{bmatrix} \Gamma^2 + \Sigma^2 & 2\Gamma\Sigma & 0 \\ 2\Gamma\Sigma & \Gamma^2 + \Sigma^2 & 0 \\ 0 & 0 & I \end{bmatrix} \begin{bmatrix} V_{11} & 0 \\ 0 & V_{22} \end{bmatrix}.$$

The above equality implies that every $\lambda$ which satisfy

$$\Big(\lambda I - (\Gamma^2 + \Sigma^2)\Big)^2 - 4\Gamma^2\Sigma^2 = 0 \tag{2.16}$$

is eigenvalue of $F^*F$ possible different from one, and there is at least $n - 2l$ eigenvalues of $F^*F$ equal to one.

Equation (2.16) written component wise has the form

$$\begin{gathered} \Big(\lambda_i - (\gamma_i^2 + \sigma_i^2)\Big)^2 - 4\gamma_i^2\sigma_i^2 = 0, \quad \text{or} \\ \lambda_i^2 - 2\lambda_i(\gamma_i^2 + \sigma_i^2) + (\gamma_i^2 - \sigma_i^2) = 0, \end{gathered}$$

where $i = 1, \dots, l$. Thus, solutions $(\lambda_i)_{1,2}$ of upper equation a given by

$$(\lambda_i)_{1,2} = (\gamma_i \pm \sigma_i)^2, \tag{2.17}$$

where $i = 1, \dots, l$. From (2.17) follows the first two equality from (2.15). ∎

From (2.15) follows the well known fact that singular values of $J$-unitary matrix $F$, different from one, come in reciprocital pairs. Indeed,

$$\tau_i = \gamma_i + \sigma_i = \frac{\gamma_i^2 - \sigma_i^2}{\gamma_i - \sigma_i} = \frac{1}{\tau_{l+i}} \quad i = 1, \dots, l.$$

Further, we show how one can easily derive a norm of the matrix $F$ using the upper results. More precisely, we shall drive the results of property 4. from the introduction in the section 2.1.

Let $F = \begin{bmatrix} F_+ & F_- \end{bmatrix}$ be $J$ orthogonal matrix where columns of $F_+$, $F_-$ spanned positive, negative subspace, respectively, that is, $F_+^* J F_+ = I_l$ and $F_-^* J F_- = I_{n-l}$. From (2.4) follows that

$$\|F\| = \gamma_1 + \sigma_1 = \gamma_1 + \sqrt{\gamma_1^2 - 1}, \tag{2.18}$$

or

$$\|F\| = \sigma_1 + \sqrt{1 + \sigma_1^2}, \tag{2.19}$$

where $\gamma_1$ is the greatest diagonal element of the matrix $\Gamma$ and $\sigma_1$ is the greatest diagonal element of the matrix $\Sigma$ in Theorem 2.1. Further, if we write

$$F = \begin{bmatrix} F_{11} & F_{12} \\ F_{21} & F_{22} \end{bmatrix} = \begin{bmatrix} F_+(1:l, 1:l) & F_-(1:l, l+1:n) \\ F_+(l+1:n, 1:l) & F_+(l+1:n, l+1:n) \end{bmatrix}, \tag{2.20}$$

again from the Theorem 2.1 follows that

$$\|F_{11}\| = \|F_{22}\| = \gamma_1, \qquad \|F_{12}\| = \|F_{21}\| = \sigma_1.$$

Now, from (2.18) follows that we can derive a norm of matrix $F$ knowing only one of blocks in (2.20). For example

$$\|F\| = \|F_{11}\| + \sqrt{\|F_{11}\|^2 - 1}. \tag{2.21}$$

Also, note that matrices whose columns spanned positive and negative subspaces have the same norm, that is,

$$\|F_+\| = \|F_-\| = \gamma_1 + \sigma_1.$$

In the following example we will illustrate the upper results on the $J$-unitary matrix $F$, where $J = \operatorname{diag}(1, -1, -1, -1)$. The indefinite scalar product generated by $J$ of this form, has been considered in [3] for problems in linear optics.

**Example 2.1** *Let* $J = \operatorname{diag}(1, -1, -1, -1)$*, and let* $F$ *given by*

$$F = \begin{bmatrix} 1.9266 & 0.4599 & 1.5616 & -0.2485 \\ 0.5681 & 1.0092 & 0.4536 & 0.3138 \\ 0.3806 & -0.3282 & 0.6860 & 0.7527 \\ -1.4981 & -0.2921 & -1.6620 & 0.6298 \end{bmatrix}$$

*be $J$-unitary matrix. Since $F_{11} = 1.9266$, from (2.21) follows that*

$$\|F\| = 1.9266 + \sqrt{1.9266^2 - 1} = 3.5734.$$

The equation (2.21) shows that sometimes, when $J$ has special structure (small number of positive elements or vice versa), we can easily compute a norm of $J$-unitary matrix $F$ knowing only one element $F(1,1)$ (or principal submatrix $F_{11}$) of $F$.

### 2.3.2 Computing the norm of $J$-unitary matrix

As we have already mentioned in Section 1.2, the difference between eigenvalue bounds for positive definite and indefinite Hermitian matrices, when such a difference exists, is given by additional factor $\|F\|^2$, where $F$ is $J$-unitary. This matrix $F$ is obtained by the hyperbolic singular value decomposition (HSVD) of the pair $(G, J)$, described in [35]. We show how to compute $\|F\|$ knowing only positive (or negative) eigenvalues and corresponding eigenvectors of the matrix $GJG^*$, which can be successfully applied in the case when $J$ has only one positive or one negative diagonal element. Also, we will present the bound by Slapničar and Veselić for $\|F\|^2$ [42].

First let us define HSVD. Let $G$ be $n \times m$ matrix, where $n \leq m$, such that $H = GJG^*$ is $n \times n$ non-singular. By [35], there exists an $n \times n$ unitary matrix $U$ and $m \times m$ matrix $F$, such that

$$F^*JF = \widehat{J} \equiv \begin{bmatrix} \widehat{J}_1 & \\ & \widehat{J}_2 \end{bmatrix}, \tag{2.22}$$

and

$$G = U\Sigma F^*. \tag{2.23}$$

Here $\widehat{J}$ is an $n \times n$ diagonal matrix with $\pm 1$ on its diagonal, possibly different from $J$ (that is, $\widehat{J}$ is permutation of $J$) and $\Sigma$ is $n \times m$ diagonal matrix with positive diagonal entries.

Let

$$H \;=\; U\Sigma\widehat{J}\Sigma^*U^*, \tag{2.24}$$

be the eigenvalue decomposition of the matrix $H = GJG^*$, where $\Lambda = \Sigma\widehat{J}\Sigma^*$ is a diagonal matrix whose entries are the eigenvalues of $H$. Further, we can express $\Lambda$ as

$$\Lambda = |\Lambda|^{1/2}\widehat{J}|\Lambda|^{1/2},$$

where $\widehat{J}$ is $n$-dimensional diagonal matrix whose diagonal entries are the signs of the eigenvalues.

Thus, we can write $\Sigma$ as

$$\Sigma \;=\; [\,|\Lambda|^{1/2} \quad 0\,]\,. \tag{2.25}$$

To establish (2.23) we have to construct non-singular matrix $F$ which satisfies

$$G^*G = F\Sigma^*U^*U\Sigma F^* = F\begin{bmatrix} |\Lambda|^{1/2} & 0 \\ 0 & 0 \end{bmatrix}F^*. \tag{2.26}$$

If we set $F = \begin{bmatrix} F_1 & F_2 \end{bmatrix}$, where $F_1$ is defined by

$$F_1 = G^*U|\Lambda|^{-1/2}, \tag{2.27}$$

then (2.26) is satisfied. Further, from (2.22) it follows that the matrix $F$ satisfies

$$F_1^*JF_1 \;=\; \widehat{J}_1 \tag{2.28}$$

$$F_2^*JF_1 \;=\; 0 \tag{2.29}$$

$$F_2^*JF_2 \;=\; \widehat{J}_2. \tag{2.30}$$

(2.28) follows directly from (2.24), (2.25) and (2.27). If we choose a matrix $F_2$ so that the columns of $F_2$ span $(JF_1)^\perp$, that is, $\mathcal{R}(F_2) \subseteq \mathcal{R}((JF_1)^\perp)$, then (2.29) will be satisfied as well.

Let the columns of $Q_0$ form the orthonormal basis for $(JF_1)^\perp$, that is, $Q_0^*JF_1 = 0$. Further, let

$$Q_0^*JQ_0 \equiv U_q\Omega_{n-r}U_q^* = U_q|\Omega_{n-r}|^{1/2}\widehat{J}_2|\Omega_{n-r}|^{1/2}U_q^*$$

be the eigenvalue decomposition of $Q_0^*JQ_0$. Define $F_2$ as

$$F_2 = Q_0U_q|\Omega_{n-r}|^{-1/2}.$$

It follows that $F_2^*JF_2 = \widehat{J}_2$ which shows that $F_2$ satisfies (2.29). It is not complicated to show that $F_2$ also satisfies (2.30) (see [35]).

Note that $F$ is not $J$-unitary matrix in the sense of our definition, because $F^*JF \neq J$. But, when $G$ is non-singular $n \times n$ matrix, then using the fact that $\widehat{J}$ is permutation of $J$ we can permute columns of $F$ so that $F^*JF = J$, and still

$$G = U\Sigma F^*,$$

here columns of $U$ and $\Sigma$ are also permuted accordingly. Also, note that diagonal elements of $\Sigma$ are square roots of the absolute values of the eigenvalues of matrix $H = GJG^*$. Thus, hyperbolic singular value decomposition of a square matrix $G$ can be written as

$$G = U|\Lambda|^{1/2}F^*, \qquad F^*JF = J, \tag{2.31}$$

where $\Lambda = |\Lambda|^{1/2}J|\Lambda|^{1/2}$ is diagonal matrix with eigenvalues of $H = GJG^*$ on its diagonal.

Now (2.21) implies in order to compute $\|F\|$ it is enough to know positive (or negative) eigenvalues and corresponding eigenvectors of matrix $H$. If we write (2.31) as

$$\begin{bmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{bmatrix} = \begin{bmatrix} U_{11} & U_{12} \\ U_{21} & U_{22} \end{bmatrix} \begin{bmatrix} |\Lambda_{11}|^{1/2} & 0 \\ 0 & |\Lambda_{22}|^{1/2} \end{bmatrix} \begin{bmatrix} F_{11}^* & F_{12}^* \\ F_{21}^* & F_{22}^* \end{bmatrix}, \tag{2.32}$$

where $G_{11}$, $U_{11}$, $\Lambda_{11}$ and $F_{11}$ are $l \times l$ dimensional submatrices. Then we have the following theorem

**Theorem 2.3** *Let $F$ be a $J$-unitary matrix obtained by HSVD of a non-singular matrix $G$ as in (2.32), where $J = I_l \oplus (-I_{n-l})$. Then*

$$\begin{aligned} \|F\| \;\; = \;\; & \sqrt{\|(G_{11}^*U_{11} + G_{21}^*U_{21})|\Lambda_{11}|^{-1/2}\|^2 - 1} + \\ & \|(G_{11}^*U_{11} + G_{21}^*U_{21})|\Lambda_{11}|^{-1/2}\|. \end{aligned} \tag{2.33}$$

**Proof.** From (2.31) follows that

$$G^*U = F|\Lambda|^{1/2}$$

which together with (2.32) gives

$$\begin{bmatrix} F_{11} \\ F_{21} \end{bmatrix} = \begin{bmatrix} (G_{11}^*U_{11} + G_{21}^*U_{21})|\Lambda_{11}|^{-1/2} \\ (G_{12}^*U_{11} + G_{22}^*U_{21})|\Lambda_{11}|^{-1/2} \end{bmatrix}.$$

Now, using (2.21) we obtain (2.33). ■

The above result gives us one possible way to compute $\|F\|$, but often we do not have enough information to do so. Thus, often it is important to know the upper bound for $\|F\|$.

Slapničar and Veselić [42] have shown that:

$$\kappa(F) = \|F\|^2 \le \min_{\bar{D}} \sqrt{\kappa(\bar{D}^*G^*G\bar{D})}, \tag{2.34}$$

where $\bar{D}J = J\bar{D}$. One possible choice for $\bar{D}$ for which the minimum is attained is

$$\bar{D} = \bar{H}_{11}^{1/2} \oplus \bar{H}_{22}^{1/2},$$

where $\bar{H} = G^*G$. If $\bar{D}$ is diagonal so that $A_{ii} = 1$, then

$$\|F\|^2 \le \sqrt{\kappa(A)}. \tag{2.35}$$

### 2.3.3 Bounds for hyperbolic sine of the maximal hyperbolic canonical angle

In this section we present one more application of Theorem 2.1 and Corollary 2.1. We will derive a bound for $\|F\|$ in terms of $\|V\|$, where $F$ and $V$ are two $J$-unitary matrices. Our second result is bound for $\|V_2^* J F_1\|$ in terms of the maximal hyperbolic sine of the hyperbolic canonical angle between positive subspaces $\mathcal{R}(V_+)$ and $\mathcal{R}(F_+)$ when $V_2 \neq V_-$ and $F_1 \neq F_+$.

Let us state some well known facts about canonical angle between two subspaces. Let $\mathcal{X}$ and $\mathcal{Y}$ be two subspaces of $\mathbf{C}^n$ of the same dimension, and let $X$ and $Y$ be their orthonormal basis, respectively. Further, let $X_\perp$ be orthonormal basis for subspace $\mathcal{X}_\perp$, which is orthogonal onto $\mathcal{X}$. Then, the sine of the greatest canonical angle $\sin\alpha$ between subspaces $\mathcal{X}$ and $\mathcal{Y}$ is given by [46, Corollary I.5.4.]

$$\sin\alpha = \|X_\perp^* Y\|.$$

From Corollary 2.1 follows that similar result holds for the hyperbolic sine of the maximal hyperbolic canonical angle between two positive (or negative) subspaces spanned by columns of $J$-unitary matrices. Indeed, let $V = \begin{bmatrix} V_+ & V_- \end{bmatrix}$ and $F = \begin{bmatrix} F_+ & F_- \end{bmatrix}$ be two $J$-unitary matrices, as in Corollary 2.1 and let $\sinh\alpha_1$ be hyperbolic sine of the maximal hyperbolic canonical angles between subspaces $\mathcal{R}(V_+)$ and $\mathcal{R}(F_+)$, defined by Definition 2.1. Then from Corollary 2.1 follows that

$$\sinh\alpha_1 = \|F_-^* J V_+\| \equiv \sigma_1. \tag{2.36}$$

Now, using this and (2.19) we can bound $\|F\|$ in terms of $\|V\|$, which will be presented in the following theorem.

**Theorem 2.4** *Let $V = \begin{bmatrix} V_+ & V_- \end{bmatrix}$ and $F = \begin{bmatrix} F_+ & F_- \end{bmatrix}$ be two $J$-unitary matrices, where $V_+, F_+$ and $V_-, F_-$ are defined similarly as in Corollary 2.1. Then*

$$\|F\| \leq \left(\|F_-^* J V_+\| + \sqrt{1 + \|F_-^* J V_+\|^2}\right)\|V\|. \tag{2.37}$$

**Proof.** From (2.36) and (2.19) we have

$$\begin{aligned} \|F^* J V\| &= \|F_-^* J V_+\| + \|F_+^* J V_+\| \\ &= \|F_-^* J V_+\| + \sqrt{1 + \|F_-^* J V_+\|^2}. \end{aligned}$$

This and the fact that $\|V^{-1}\| = \|V\|$ (this is the property 1. from page 34) implies

$$\frac{1}{\|V\|} \cdot \|F\| \quad \leq \quad \|F_-^* J V_+\| + \sqrt{1 + \|F_-^* J V_+\|^2},$$

which implies (2.38). ■

Unfortunately, the relation $\|F_-^* J V_+\| = \sinh \alpha_1$ holds only for positive and negative subspaces. In general when two $J$-unitary matrices $V$ and $F$ are decomposed as $V = \begin{bmatrix} V_1 & V_2 \end{bmatrix}$ and $F = \begin{bmatrix} F_1 & F_2 \end{bmatrix}$, where $V_2 \neq V_-$ and $F_1 \neq F_+$, then

$$\|F_2^* J V_1\| \neq \|F_1^* J V_2\|.$$

This fact is illustrated by the following example.

**Example 2.2** *Let $F$ and $V$ be $J$-unitary matrices with* $J = \operatorname{diag}(1, 1, -1, -1)$.

$$F = \begin{bmatrix} 3.4308 & -0.2029 & -2.4125 & -2.2342 \\ 0.5273 & 0.9992 & -0.2264 & -0.4745 \\ -3.3194 & 0.0609 & 2.5591 & 2.3395 \\ -0.1732 & -0.1892 & -0.5680 & 0.8621 \end{bmatrix},$$

$$V = \begin{bmatrix} 1.0376 & -0.7290 & -0.7659 & -0.1466 \\ 0.7020 & 0.8002 & -0.2416 & 0.2735 \\ -0.6495 & 0.0569 & 1.0447 & -0.5778 \\ -0.3841 & 0.4107 & 0.7441 & 0.8732 \end{bmatrix}.$$

*Then, rounded to four digits,*

$$\|F_-^* J V_+\| = \|F_+^* J V_-\| = 2.334.$$

*However,* $F_1 = F(:,1)$, $F_2 = F(:,2:4)$, $V_1 = V(:,1)$ *and* $V_2 = V(:,2:4)$, *and we have*

$$\|F_2^* J V_1\| = 1.5279 \qquad \|F_1^* J V_2\| = 2.9215.$$

Fortunately, from Theorem 2.1 and Corollary 2.1 we can derive a bound for $\|F_2^* J V_1\|$ in the terms of hyperbolic angles between positive subspaces $\mathcal{R}(V_+)$ and $\mathcal{R}(F_+)$.

**Corollary 2.3** *Let* $V = \begin{bmatrix} V_1 & V_2 \end{bmatrix} = \begin{bmatrix} V_+ & V_- \end{bmatrix}$ *and* $F = \begin{bmatrix} F_1 & F_2 \end{bmatrix} = \begin{bmatrix} F_+ & F_- \end{bmatrix}$ *be two* $J$*-unitary matrices, where* $\mathcal{R}(V_1)$ *and* $\mathcal{R}(F_1)$ *are* $k$*-dimensional subspaces of* $\mathbf{C}^n$*. Also, let* $\mathcal{R}(V_+)$ , $\mathcal{R}(F_+)$, $\mathcal{R}(V_-)$ *and* $\mathcal{R}(F_-)$ *be positive and negative subspaces corresponding to the matrix* $J$*, respectively, defined similarly as in Corollary 2.1. Let* $U_{11}, U_{22}, V_{11}$ *and* $V_{22}$ *be the unitary matrices and* $\Gamma$ *and* $\Sigma$ *be the diagonal matrices defined in Theorem 2.1 such that (2.4) holds for* $J$*-unitary matrix* $F^*JV$*. Then, we have the following bounds*

$$\|F_2^* J V_1\| \leq \|\Sigma\| + \|\Gamma - U_{11}V_{11}^*\|, \quad \text{for} \quad k < l \tag{2.38}$$

*and*

$$\|F_2^* J V_1\| \leq \|\Sigma\| + \|\mathrm{diag}\,(\Gamma, I_{2n-l}) + U_{22}V_{22}^*\| \text{ for } k > l. \tag{2.39}$$

**Proof**. First, we consider the case when $k < l$. Note that for $k < l$ holds

$$\mathcal{R}(V_1) \subset \mathcal{R}(V_+), \mathcal{R}(V_-) \subset \mathcal{R}(V_2) \text{ and } \mathcal{R}(F_1) \subset \mathcal{R}(F_+), \mathcal{R}(F_-) \subset \mathcal{R}(F_2).$$

Let $V_+ = \begin{bmatrix} V_1^+ & V_2^+ \end{bmatrix}$ and $F_+ = \begin{bmatrix} F_1^+ & F_2^+ \end{bmatrix}$ such that $V_1 = V_1^+$ and $V_2 = \begin{bmatrix} V_2^+ & V_- \end{bmatrix}$, and similarly $F_1 = F_1^+$ and $F_2 = \begin{bmatrix} F_2^+ & F_- \end{bmatrix}$. Our aim is to derive a bound for $\|F_2^* J V_1\|$. We have

$$\|F_2^* J V_1\| = \| \begin{bmatrix} F_2^+ & F_- \end{bmatrix}^* J V_1^+ \| \leq \|(F_2^+)^* J V_1^+\| + \|F_-^* J V_1^+\|. \tag{2.40}$$

From Corollary 2.1 we have

$$\|F_-^* J V_1^+\| \leq \sigma_1,$$

which gives a bound for the second term in the left hand side of (2.40). Further, let $T_+$ be such a matrix that $F_+ = V_+ + T_+$. Now, from $V_+^* J V_+ = I_l$ and from

$$F_+^* J V_+ = U_{11}^* \Gamma V_{11},$$

( see the proof of Theorem 2.1), we have

$$T_+^* J V_+ = U_{11}^* \Gamma V_{11} - I_l.$$

Further, note that if we write $F_2^+ = V_2^+ + T_2^+$, then

$$(F_2^+)^* J V_1^+ = (T_2^+)^* J V_1^+,$$

and we are able to bound the first term in the left hand side of (2.40). Indeed,

$$\|(F_2^+)^* J V_1^+\| = \|T_+^* J V_+\| \leq \|\Gamma - U_{11}V_{11}^*\|.$$

Now we can write the bound (2.40) as

$$\|F_2^* J V_1\| \;\leq\; \|\Sigma\| + \|\Gamma - U_{11}V_{11}^*\|. \tag{2.41}$$

This completes the proof of (2.38).

Similarly, in the case when $k > l$. First, note that for $k > l$ holds

$$\mathcal{R}(V_+) \subset \mathcal{R}(V_1), \mathcal{R}(V_2) \subset \mathcal{R}(V_-) \text{ and } \mathcal{R}(F_+) \subset \mathcal{R}(F_1), \mathcal{R}(F_2) \subset \mathcal{R}(F_-).$$

Let $V_- = \begin{bmatrix} V_1^- & V_2^- \end{bmatrix}$ and $F_- = \begin{bmatrix} F_1^- & F_2^- \end{bmatrix}$ such that $V_1 = \begin{bmatrix} V_+ & V_1^- \end{bmatrix}$, $V_2 = V_2^-$ and $F_1 = \begin{bmatrix} F_+ & F_1^- \end{bmatrix}$, $F_2 = F_2^-$. Thus we have

$$\|F_2^* J V_1\| \leq \|(F_2^-)^* J V_+\| + \|(F_2^-)^* J V_1^-\|. \tag{2.42}$$

Using a similar arguments as in the case $k < l$, we can show that the following bounds hold:

$$\begin{aligned} \|(F_2^-)^* J V_1^-\| &\leq \|\mathrm{diag}\,(\Gamma, I_{n-l}) + U_{22}V_{22}^*\|, \\ \|(F_2^-)^* J V_+\| &\leq \sigma_1. \end{aligned}$$

The bound (2.42) now has the following form

$$\|F_2^* J V_1\| \;\leq\; \sigma_1 + \|\mathrm{diag}\,(\Gamma, I_{n-l}) - U_{22}^* V_{22}\|, \tag{2.43}$$

which completes the proof. ■

The following examples illustrates the quality of bounds (2.38) and (2.39). First we consider two arbitrary matrices.

**Example 2.3** *Let $F$ and $V$ be two $J$-unitary matrices from the Example 2.2, and $J =$ $\mathrm{diag}\,(1, 1, -1, -1)$. The CS-decomposition of the matrix $F_0 = F^* J V$ gives us*

$$F_0 = UPV^*,$$

*where $U$ and $V$ are unitary matrices,*

$$U = \begin{bmatrix} 0.9792 & -0.2030 & 0 & 0 \\ -0.2030 & -0.9792 & 0 & 0 \\ 0 & 0 & -0.8 & -0.4689 \\ 0 & 0 & -0.4689 & 0.8832 \end{bmatrix},$$

$$V = \begin{bmatrix} 0.6218 & -0.7832 & 0 & 0 \\ -0.7832 & -0.6218 & 0 & 0 \\ 0 & 0 & 0.3538 & -0.9353 \\ 0 & 0 & -0.9353 & -0.3538 \end{bmatrix},$$

*and $P$ is given by*

$$P = \begin{bmatrix} 2.5393 & 0 & 2.3344 & 0 \\ 0 & 1.0150 & 0 & 0.1734 \\ 2.3341 & 0 & 2.5393 & 0 \\ 0 & 0.1734 & 0 & 1.0150 \end{bmatrix}.$$

*If we consider the case $F_1 = F(:,1)$, $F_2 = F(:,2:4)$ and $V_1 = V(:,1)$, $V_2 = V(:,2:4)$, then the bounds for $\|F_2^* J V_1\| = 1.5279$ and $\|F_1^* J V_2\| = 2.9215$ are*

$$\|F_2^* J V_1\| \leq 4.2917 \quad \text{and} \quad \|F_1^* J V_2\| \leq 5.2425, \tag{2.44}$$

respectively. The reason why (2.44) is not too sharp is in the fact that $F$ and $V$ are two arbitrary $J$-orthogonal matrices and subspaces spanned by its columns are not close.

The next example considers a case when $F$ and $V$ are close (that is, when $V$ is a perturbation of $F$).

**Example 2.4** *Let $F$ and $V$ be two $J$-unitary matrices*

$$F = \begin{bmatrix} 1.2292 & -0.1693 & 0.6952 & 0.2370 \\ 0.0474 & 1.1128 & -0.3153 & 0.3758 \\ 0.6793 & -0.0568 & 0.9665 & 0.7284 \\ -0.2274 & 0.5136 & -0.8054 & 0.8166 \end{bmatrix},$$

$$V = \begin{bmatrix} 1.2280 & -0.1681 & 0.6928 & 0.2370 \\ 0.0476 & 1.1124 & -0.3140 & 0.3756 \\ 0.6775 & -0.0556 & 0.9652 & 0.7283 \\ -0.2261 & 0.5125 & -0.8043 & 0.8166 \end{bmatrix},$$

*and $J = \operatorname{diag}(1,1,-1,-1)$. The CS-decomposition of the matrix $F_0 = F^* J V$ gives us*

$$F_0 = UPV^*,$$

*where $U$ and $V$ are unitary matrices*

$$U = \begin{bmatrix} 0.8332 & -0.5529 & 0 & 0 \\ -0.5529 & -0.8332 & 0 & 0 \\ 0 & 0 & -0.9990 & 0.0449 \\ 0 & 0 & 0.0449 & 0.9990 \end{bmatrix},$$

$$V = \begin{bmatrix} 0.8331 & -0.5532 & 0 & 0 \\ -0.5532 & -0.8331 & 0 & 0 \\ 0 & 0 & -0.9990 & 0.0448 \\ 0 & 0 & 0.0448 & 0.9990 \end{bmatrix},$$

*and $P$ is given by*

$$P = \begin{bmatrix} 1.0000 & 0.0000 & -0.0022 & 0.0000 \\ 0.0000 & 1.0000 & 0.0000 & -0.0001 \\ 0.0022 & 0.0000 & -1.0000 & 0.0000 \\ 0.0000 & 0.0001 & 0.0000 & -1.0000 \end{bmatrix}.$$

*If we consider the case $F_1 = F(:,1)$,$F_2 = F(:,2:4)$ and $V_1 = V(:,1)$, $V_2 = V(:,2:4)$, then the bounds for $\|F_2^* J V_1\| = 0.001864$ and $\|F_1^* J V_2\| = 0.001865$ are*

$$\|F_2^* J V_1\| \leq 0.00253, \quad \text{and} \quad \|F_1^* J V_2\| \leq 0.00228,$$

respectively.

## 2.4 Bounds for the norm of $J$-unitary matrices

As we will see later, it is of our interest to say something about relation between $\|\tilde{F}\|$ and $\|F\|$, where $\tilde{F}$ and $F$ are $J$-unitary singular vector matrices obtained by HSVD of quadratic matrix $G \in \mathbf{C}^{n\times n}$ and $\tilde{G} \in \mathbf{C}^{n\times n}$, respectively. Thus, in this section we present two bounds for $\|\tilde{F}\|$ in terms of $\|F\|$. We consider two kinds of perturbations of matrix $G$:

1. $\tilde{G} = G\,(I + E_R)$,
2. $\tilde{G} = (I + E_L)\,G$.

As we will see, in both cases, the bound for $\|\tilde{F}\|$, depends only on $\|F\|$ and norm of error $\|E_R\|$ or $\|E_L\|$.

Let

$$G = U\Sigma F^{-1}, \qquad \tilde{G} = \tilde{U}\tilde{\Sigma}\tilde{F}^{-1}, \tag{2.45}$$

be HSVD of $G$ and $\tilde{G}$, respectively (as described in section 2.3.2). Here $U$ and $\tilde{U}$ are unitary, $F$ and $\tilde{F}$ are $J$-unitary, that is,

$$F^* J F = J, \qquad \tilde{F}^* J \tilde{F} = J,$$

and $\Sigma$ and $\tilde{\Sigma}$ are diagonal matrices with positive elements on the diagonals. Here, we also assume that $J = I_l \oplus (-I_{n-l})$.

The first lemma contains the bound for $\|\tilde{F}\|$ in terms of $\|F\|$ and $\|E_R\|$, when $G$ is perturbed to $\tilde{G} = G\,(I + E_R)$.

**Lemma 2.1** *Let $G$ and $\tilde{G} = G\,(I + E_R)$ be two $n \times n$ non-singular matrices. Let $F$ and $\tilde{F}$ be $J$-unitary matrices obtained by hyperbolic singular value decomposition of $G$ and $\tilde{G}$, respectively, as in (2.45), where $J$ is given by $J = I_l \oplus (-I_{n-l})$. Define $\gamma = \|E_R\|_F/(1 - \|E_R\|)$. If*

$$\gamma \|F\|^2 < \frac{1}{4}, \tag{2.46}$$

*then*

$$\|\tilde{F}\| \leq \frac{\|F\|}{\sqrt{1 - 4\gamma\|F\|^2}}. \tag{2.47}$$

**Proof**. Let $F$ and $\tilde{F}$ be partitioned in block columns as

$$F = [\, F_+ \quad F_- \,] \qquad \text{and} \qquad \tilde{F} = [\, \tilde{F}_+ \quad \tilde{F}_- \,].$$

$F_+$ and $\tilde{F}_+$ span the positive subspaces with respect to $J$, respectively, since $F_+^* J F_+ = I_l$ and $\tilde{F}_+^* J \tilde{F}_+ = I_l$. Similarly, $F_-$ and $\tilde{F}_-$ spann the negative subspaces, respectively.

From (2.45) follows that $F$ and $\tilde{F}$ diagonalize the pairs $(G^*G, J)$ and $(\tilde{G}^*\tilde{G}, J)$, respectively. Moreover, we can write

$$\begin{aligned} \begin{bmatrix} F_+^* \\ F_-^* \end{bmatrix} G^*G\,[\, F_+ \quad F_- \,] &= \begin{bmatrix} \Sigma_+^2 & \\ & \Sigma_-^2 \end{bmatrix}, \\ \begin{bmatrix} F_+^* \\ F_-^* \end{bmatrix} J\,[\, F_+ \quad F_- \,] &= \begin{bmatrix} I_l & \\ & -I_{n-l} \end{bmatrix}, \end{aligned} \tag{2.48}$$

where $\Sigma_+$ and $\Sigma_-$ are diagonal positive definite matrices which are partitioned according to $J$ and $F$. Relations analogous to (2.48) hold for perturbed quantities.

$J$-unitarity of $F$ and (2.48) imply

$$G^*GF_+ = JF_+\Sigma_+^2 \qquad \text{and} \qquad G^*GF_- = -JF_-\Sigma_-^2. \tag{2.49}$$

Again, relations analogous to (2.49) hold for $\tilde{G}^*\tilde{G}$, $\tilde{F}$, $\tilde{\Sigma}_+^2$ and $\tilde{\Sigma}_-^2$, as well. By premultiplying $\tilde{G}^*\tilde{G}\tilde{F}_+ = J\tilde{F}_+\tilde{\Sigma}_+^2$ by $F_-^*$ we have, after using (2.49) and rearranging,

$$\Sigma_-^2 F_-^* J \tilde{F}_+ + F_-^* J \tilde{F}_+ \tilde{\Sigma}_+^2 = F_-^*(\tilde{G}^*\tilde{G} - G^*G)\tilde{F}_+. \tag{2.50}$$

Set $\widehat{E}_R = E_R^*(I+E_R)^{-*}$ (the inverse exists since (2.46) implies $\|E_R\| < 1$). By using the identity $\tilde{G}^*\tilde{G} - G^*G = \widehat{E}_R\tilde{G}^*\tilde{G} + G^*GE_R$, the above equality can be written component-wise as

$$\begin{aligned}[F_-^* J\tilde{F}_+]_{ij}([\Sigma_-^2]_{ii} + [\tilde{\Sigma}_+^2]_{jj}) &= [F_-]_{:i}^* \left(\widehat{E}_R\tilde{G}^*\tilde{G} + G^*GE_R\right)[\tilde{F}_+]_{:j} \\ &= [F_-]_{:i}^*\widehat{E}_R J[\tilde{F}_+]_{:j}[\tilde{\Sigma}_+^2]_{jj} - [\Sigma_-^2]_{ii}[F_-]_{:i}^* JE_R[\tilde{F}_+]_{:j}.\end{aligned}$$

Here $[F]_{:k}$ denotes the $k$-th column of the matrix $F$. By dividing this equality by $[\Sigma_-^2]_{ii} + [\tilde{\Sigma}_+^2]_{jj}$ and by using the fact that

$$\frac{\max\{[\Sigma_-^2]_{ii}, [\tilde{\Sigma}_+^2]_{jj}\}}{[\Sigma_-^2]_{ii} + [\tilde{\Sigma}_+^2]_{jj}} < 1,$$

we have

$$|[F_-^* J\tilde{F}_+]_{ij}| \le |[F_-]_{:i}^*\widehat{E}_R J[\tilde{F}_+]_{:j}| + |[F_-]_{:i}^* JE_R[\tilde{F}_+]_{:j}|.$$

Since $\|\widehat{E}_R\|_F \le \gamma$, by taking Frobenius norm we have

$$\|F_-^* J\tilde{F}_+\|_F \le \|F_-\|\,\|\tilde{F}_+\|(\|\widehat{E}_R\|_F + \|E_R\|_F) \le 2\gamma\|F\|\,\|\tilde{F}\|.$$

By inserting this inequality in (2.37), we have

$$\|\tilde{F}\| \le \left(2\gamma\|F\|\,\|\tilde{F}\| + \sqrt{1 + (2\gamma\|F\|\,\|\tilde{F}\|)^2}\right)\|F\|.$$

After a simple manipulation we obtain

$$\|\tilde{F}\|^2 \le \frac{\|F\|^2}{1 - 4\gamma\|F\|^2},$$

as desired. ∎

Note that, the magnitude of the change in $\tilde{F}$ depends only on $\|E_R\|$ and does not depend on matrix $G$.

Now we present our second result about norm of perturbed $J$-unitary matrix $F$. The following lemma contains the bound for $\|\tilde{F}\|$ when $G$ is perturbed to $\tilde{G} = (I + E_L)G$.

**Lemma 2.2** *Let $G$ and $\tilde{G} = (I + E_L)G$ be two $n \times n$ non-singular matrices. Let $F$ and $\tilde{F}$ be $J$-unitary matrices obtained by hyperbolic singular value decomposition of $G$ and $\tilde{G}$, respectively, as in (2.45), where $J = I_l \oplus (-I_{n-l})$. Define $\eta = \|E_L\|_F/(1 - \|E_L\|)$. If $\eta < 1$, then*

$$\|\tilde{F}\| \le \left(\eta + \sqrt{1 + \eta^2}\right)\|F\|. \tag{2.51}$$

**Proof**. Let $F$ and $\tilde{F}$ be partitioned in block columns as

$$F = [\, F_+ \quad F_- \,] \qquad \text{and} \qquad \tilde{F} = [\, \tilde{F}_+ \quad \tilde{F}_- \,],$$

as in Lemma 2.1. Further, we assume that the relation (2.48) and (2.49) from Lemma 2.1 hold here, as well. Note that

$$\begin{aligned} \tilde{G}^*\tilde{G} - G^*G &= G^*(E_L + E_L^*(I + E_L))G \\ &= G^*(E_L(I + E_L)^{-1} + E_L^*)\tilde{G}. \end{aligned}$$

Here the inverse exists since $\eta < 1$ implies $\|E_L\| < 1$. Using this identity the relation (2.50) can be written component-wise as

$$\begin{aligned} [F_-^* J\tilde{F}_+]_{ij} \left([\Sigma_-^2]_{ii} + [\tilde{\Sigma}_+^2]_{jj}\right) &= [F_-]_{:i}^* G^*(E_L(I + E_L)^{-1} + E_L^*)\tilde{G}[\tilde{F}_+]_{:j} \\ &= [\Sigma_-]_{ii} U_{:i}^* \left(E_L(I + E_L)^{-1} + E_L^*\right) \tilde{U}_{:j} [\tilde{\Sigma}_+]_{jj}. \end{aligned} \tag{2.52}$$

Here we have used (2.45), that is,

$$G[F_-]_{:i} = [\Sigma_-]_{ii} U_{:i}^*, \qquad \tilde{G}[\tilde{F}_+]_{:j} = [\Sigma_+]_{jj}\tilde{U}_{:j},$$

where $U_{:k}$ denotes the $k$-th column of the unitary matrix $U$. By dividing (2.52) by $[\Sigma_-^2]_{ii} + [\tilde{\Sigma}_+^2]_{jj}$ and by using the fact that

$$\frac{[\Sigma_-]_{ii} \cdot [\tilde{\Sigma}_+]_{jj}}{[\Sigma_-^2]_{ii} + [\tilde{\Sigma}_+^2]_{jj}} < \frac{1}{2},$$

we have

$$|[F_-^* J\tilde{F}_+]_{ij}| \le \frac{1}{2} |U_{:i} \left(E_L(I + E_L)^{-1} + E_L^*\right) \tilde{U}_{:j}|.$$

Since $\|E_L(I + E_L)^{-1} + E_L^*\|_F \le 2\,\eta$, by taking Frobenius norm we have

$$\|F_-^* J\tilde{F}_+\|_F \le \frac{1}{2}(\|E_L(I + E_L)^{-1} + E_L^*\|_F) \le \eta.$$

Using this, from (2.38) we have

$$\|F^* J\tilde{F}\|_F \le \eta + \sqrt{1 + \eta}, \tag{2.53}$$

which implies

$$\|\tilde{F}\| \le \left(\eta + \sqrt{1 + \eta^2}\right) \|F\|,$$

as desired. ∎

In the following remark we point out one possible application of the bound (2.47).

**Remark 2.1** *The following remark shows that if $J$-unitary singular vector matrices $F$ obtained by HSVD of quadratic matrix $G \in \mathbf{C}^{n\times n}$ is close to the block unitary matrix, then we can successfully apply the bound (2.47). The $F$ is close to the block unitary matrix always when $\mathcal{R}(G(:,1:l)$ is almost orthogonal to the $\mathcal{R}(G(:,l+1:n)$.*

*Let $G$ be non-singular $n\times n$ matrix, and let $J = I_l \oplus (-I_{n-l})$ be diagonal matrix with $\pm 1$ on its diagonal with $2l \le n$. Further, let*

$$G = U_0 \Sigma_0 V_0^*, \tag{2.54}$$

*be singular value decomposition of $G$, where $U_0$ is $n\times n$ and $V_0$ is $m\times m$ unitary and $\Sigma_0$ is diagonal with positive element on its diagonal. Let*

$$V_0 = \begin{bmatrix} X_{11}^* & 0 \\ 0 & X_{22}^* \end{bmatrix} \begin{bmatrix} \Gamma_V & \Sigma_V & 0 \\ -\Sigma_V & \Gamma_V & 0 \\ 0 & 0 & I \end{bmatrix} \begin{bmatrix} Y_{11} & 0 \\ 0 & Y_{22} \end{bmatrix}, \tag{2.55}$$

*be an orthogonal CS-decomposition of the unitary matrix $V_0$, where $X_{11}$ and $Y_{11}$ are $l\times l$ unitary matrices, $X_{22}$ and $Y_{22}$ are $(n-l)\times(n-l)$ unitary matrices, $\Gamma_V = \mathrm{diag}\,(\gamma_1,\ldots,\gamma_l)$, and $\Sigma_V = \mathrm{diag}\,(\sigma_1,\ldots,\sigma_l)$, with*

$$\gamma_i^2 + \sigma_i^2 = 1 \qquad i = 1,\ldots l.$$

*Also,let*

$$G = U|\Lambda|^{1/2}F^{-1},$$

*be the hyperbolic singular value decomposition of $G$.*

*Thus, we can write*

$$U_0\Sigma_0 \begin{bmatrix} X_{11}^* & 0 \\ 0 & X_{22}^* \end{bmatrix} \begin{bmatrix} \Gamma_V & \Sigma_V & 0 \\ -\Sigma_V & \Gamma_V & 0 \\ 0 & 0 & I \end{bmatrix} = U|\Lambda|^{1/2}\tilde{F}^*, \tag{2.56}$$

*where $\tilde{F}^* = F^{-1}\begin{bmatrix} Y_{11}^* & 0 \\ 0 & Y_{22}^* \end{bmatrix}$ is $J$-unitary matrix and $\|\tilde{F}\| = \|F\|$.*

*From (2.56) it follows that we can interpret the matrix*

$$\tilde{G} = U_0\Sigma_0 \begin{bmatrix} X_{11}^* & 0 \\ 0 & X_{22}^* \end{bmatrix} \begin{bmatrix} \Gamma_V & \Sigma_V & 0 \\ -\Sigma_V & \Gamma_V & 0 \\ 0 & 0 & I \end{bmatrix}$$

*as a multiplicative perturbation of the matrix*

$$\bar{G} = U_0 \Sigma_0 \begin{bmatrix} X_{11}^* & 0 \\ 0 & X_{22}^* \end{bmatrix} \tag{2.57}$$

*where* $\tilde{G} = \bar{G}(I + E_R)$, *and*

$$E_R = \begin{bmatrix} \Gamma_V & \Sigma_V & 0 \\ -\Sigma_V & \Gamma_V & 0 \\ 0 & 0 & I \end{bmatrix} - \begin{bmatrix} I & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix}.$$

*Note that (2.57) is the HSVD of* $\bar{G}$ *with the same hyperbolic singular values as singular values of* $G$, *that is, when* $E_R = 0$ *then* $\tilde{G} = G$. *By setting*

$$\varrho \equiv \|E_R\|_F / (1 - \|E_R\|), \tag{2.58}$$

*if* $\varrho < 1/4$, *then from (2.47) it follows that*

$$\|F\| \leq \frac{1}{\sqrt{1 - 4\varrho}}. \tag{2.59}$$

In [54, Theorem 5] Veselić has shown that if $G$ and $J$ are given as

$$G = [\, G_1 \quad G_2 \,], \quad J = \begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix}, \tag{2.60}$$

and if

$$\beta = \|G_1^{-1} G\| < 1, \tag{2.61}$$

then $H = GJG^*$ is positive definite (in spite of an indefinite $J$) and

$$\|F\|^2 = \frac{1 + \beta^2}{1 - \beta^2}.$$

From the SVD of the matrix $G$ (2.54), and from $CS$-decomposition of $V_0$ (2.55), after simple manipulation it follows that the condition (2.61) is equivalent to

$$\|\Sigma_V\| < \frac{1}{\sqrt{2}}.$$

In the same time, the condition $\varrho < 1/4$ from remark 2.1, implies $\|E_R\|_F < 1/5$ or $\|\Sigma_V\| < 1/\sqrt{5}$. This shows that for all matrices $G$ of the form (2.60) Remark 2.1 implies that the matrix $H = GJG^*$ is positive definite. However, if $G$ is not of the form (2.60), that is, when $H$ is not positive definite, we can not apply [54, Theorem 5], but the bound (2.59) can be applied.

# Chapter 3

# Relative perturbations of Hermitian matrices

In this chapter we consider perturbations of eigenvalues and eigenvectors of Hermitian matrices. First, we consider the case when a Hermitian matrix $H$ given in the graded form $H = D^*AD$, under additive perturbation $\tilde{H} = D^*\tilde{A}D$, where $\tilde{A} = A + \delta A$. Second, we consider the multiplicative perturbation $\tilde{H} = (I + E)^*H(I + E)$.

## 3.1 Eigenvalue perturbation bounds

### 3.1.1 Additive perturbations

In this section we consider Hermitian matrix $H$ given in the graded form $H = D^*AD$ and perturbed to $\tilde{H} = D^*(A+\delta A)D$. The main result contains eigenvalue bound which consists of two factors. The first factor essentially measures indefinitness of the matrix $H$, while the second factor measures the size of relative perturbation, and sometimes can be small regardless of matrix $D$.

The main result of this section is relative version of Hoffman and Wielandt bound [46, Theorem 3.1] for matrices given in the graded form

$$H = D^*AD, \;\; H + \delta H = D^*(A + \delta A)D, \tag{3.1}$$

where $A$ and $D$ are non-singular. Our main result, in fact, generalizes Theorem 1.3 [28, Theorem 3.2. pp. 962-963].

We use the following strategy. After introducing the eigenvalue decompositions of $A$ and its perturbation $\tilde{A}$ as

$$\begin{array}{rcl} A = Q\Delta Q^* & = & Q|\Delta|^{1/2}J|\Delta|^{1/2}Q^* \\ \tilde{A} = \tilde{Q}\tilde{\Delta}\tilde{Q}^* & = & Q|\Delta|^{1/2}(J + |\Delta|^{-1/2}Q^*\delta A Q|\Delta|^{-1/2})|\Delta|^{1/2}Q^*, \end{array} \tag{3.2}$$

we decompose both matrices in the following way:

$$\begin{array}{rcl} H & = & D^*AD = BJB^*, \qquad B = D^*Q|\Delta|^{1/2}, \\ \tilde{H} & = & D^*(A + \delta A)D = B(J + |\Delta|^{-1/2}Q^*\delta A Q|\Delta|^{-1/2})B^* \\ & \equiv & B\widehat{A}B^*, \end{array} \tag{3.3}$$

where $J$ is the inertia of $H$ and $\tilde{H}$. By Sylvester's law of inertia [20, Theorem 4.5.8] matrices $H$ and $A$ have the same inertia, which allow us to chose an eigenvalue decomposition of $A$ with the same $J$.

Then, by using hyperbolic singular value decomposition of the pairs $(B, J)$ and $\tilde{B}, J$, we derive a relative perturbation bound for the eigenvalues of $H$.

First, we describe eigenvalue decompositions of $H$ and $\tilde{H}$. Let $H = D^*AD$ be $n \times n$ non-singular indefinite Hermitian matrix, and let

$$H = U\Lambda U^*, \tag{3.4}$$

be its eigenvalue decomposition, where

$$\Lambda = \operatorname{diag}(\lambda_1, \lambda_2, \cdots, \lambda_n).$$

Let us perturb $H$ into $\tilde{H} = D^*(A+\delta A)D$, where $\delta A$ is a Hermitian perturbation. Similarly as above, let

$$\tilde{H} = \tilde{U}\tilde{\Lambda}\tilde{U}^*, \tag{3.5}$$

be the eigenvalue decomposition of $\tilde{H}$, where

$$\tilde{\Lambda} = \operatorname{diag}(\tilde{\lambda}_1, \tilde{\lambda}_2, \cdots, \tilde{\lambda}_n).$$

Further, by using (3.4) we can write

$$H = U|\Lambda|^{1/2}J|\Lambda|^{1/2}U^*,$$

where $J$ is a diagonal matrix whose diagonal elements are the signs of the eigenvalues of $H$.

We consider only perturbations such that $\|A^{-1}\|\|\delta A\| < 1$. Since

$$\eta \overset{def}{=} \||\Delta|^{-1/2}Q^*\delta AQ|\Delta|^{-1/2}\| \le \|A^{-1}\|\|\delta A\| < 1, \tag{3.6}$$

we conclude that the matrices $H$, $\tilde{H}$ and $\widehat{A}$ all have the same inertia defined by $J$. Therefore, $\widehat{A}$ can be decomposed as

$$\widehat{A} = J + N = \widehat{B}J\widehat{B}^*.$$

(In [54, Lema 6] has been -presented the similar decomposition with lower triangular factor $\widehat{B}$). Note that the assumption $\|A^{-1}\|\|\delta A\| < 1$ implies $\|NJ\| < 1$. This further insures existence of the matrix square root $(I+NJ)^{1/2}$ (see [21, Theorem 6.4.12]). Further, since $(I+NJ)^{1/2} = J\left[(I+NJ)^{1/2}\right]^* J$ we can write $\widehat{B} = (I+NJ)^{1/2}$. Since $\|NJ\| < 1$ (see [21, Theorem 6.2.8]), we can apply Taylor series of the function $(1+x)^{1/2}$ around zero,

$$(1+x)^{1/2} = 1 + \frac{1}{2}\,x - \frac{1}{8}\,x^2 + \frac{1}{16}\,x^3 - \frac{5}{128}\,x^4 \pm \cdots. \tag{3.7}$$

Therefore

$$\widehat{B} = I + \frac{1}{2}(NJ) - \frac{1}{8}(NJ)^2 + \frac{1}{16}(NJ)^3 - \frac{5}{128}(NJ)^4 \pm \cdots.$$

Similarly, by using Taylor series for the function $(1+x)^{-1/2}$ around zero,

$$(1+x)^{-1/2} = 1 - \frac{1}{2}\,x + \frac{3}{8}\,x^2 - \frac{5}{16}\,x^3 + \frac{35}{128}\,x^4 \pm \cdots,$$

the inverse $\widehat{B}^{-1}$ is defined by

$$\widehat{B}^{-1} = I - \frac{1}{2}(NJ) + \frac{3}{8}(NJ)^2 - \frac{5}{16}(NJ)^3 + \frac{35}{128}(NJ)^4 \pm \cdots.$$

We now write

$$\widehat{B} = I + T_1, \qquad \widehat{B}^{-1} = I + T_2. \tag{3.8}$$

Therefore, by using corresponding Taylor series we have

$$\begin{aligned} \|T_1\| &< \frac{1}{2}\cdot\frac{\|N\|}{\sqrt{1-\|N\|}}, \\ \|T_2\| &< \frac{1}{2}\cdot\frac{\|N\|}{(1-\|N\|)^{3/4}}, \end{aligned}$$

$$\|\widehat{B}^{-1}\| \;\leq\; \frac{1}{\sqrt{1-\|N\|}} \leq \frac{1}{\sqrt{1-\eta}}. \tag{3.9}$$

Further, (3.5) and (3.3) imply

$$\tilde{H} = \tilde{U}|\tilde{\Lambda}|^{1/2} J |\tilde{\Lambda}|^{1/2} \tilde{U}^* = B\widehat{B}J\widehat{B}^* B^* \equiv \tilde{B}J\tilde{B}^*. \tag{3.10}$$

From (3.4) and (3.3), we conclude that the hyperbolic singular value decomposition of the pair $(B, J)$ (see section 2.3.2) is given by

$$B = U|\Lambda|^{1/2} V^*, \qquad V^* J V = J, \tag{3.11}$$

where $V$ is a $J$-unitary matrix which also diagonalizes the positive definite pair $(B^*B, J)$. Similarly, from (3.10) we conclude that the hyperbolic singular value decomposition of the pair $(\tilde{B}, J)$ is given by

$$\tilde{B} = \tilde{U}|\tilde{\Lambda}|^{1/2} \tilde{V}^*, \qquad \tilde{V}^* J \tilde{V} = J. \tag{3.12}$$

Before we state our theorem we state some auxiliary results. First, we state some of the results from Li [30].

**Lemma 3.1 (R. C. Li)** *Let $a_1 \leq a_2$ and $\tilde{a}_1 \leq \tilde{a}_2$ be real numbers of the same sign, then*

$$\frac{(\tilde{a}_1 - a_1)^2}{\tilde{a}_1\, a_1} + \frac{(\tilde{a}_2 - a_2)^2}{\tilde{a}_2\, a_2} \;\leq\; \frac{(\tilde{a}_2 - a_1)^2}{\tilde{a}_2\, a_1} + \frac{(\tilde{a}_1 - a_2)^2}{\tilde{a}_1\, a_2}.$$

**Proof**. It can be verified that

$$\begin{aligned}&\frac{(\tilde{a}_1 - a_1)^2}{\tilde{a}_1\, a_1} + \frac{(\tilde{a}_2 - a_2)^2}{\tilde{a}_2\, a_2} - \frac{(\tilde{a}_2 - a_1)^2}{\tilde{a}_2\, a_1} - \frac{(\tilde{a}_1 - a_2)^2}{\tilde{a}_1\, a_2} \\ &\qquad = -\frac{(a_2 - a_1)\,(\tilde{a}_2 - \tilde{a}_1)\,(\tilde{a}_1\tilde{a}_2 + a_1 a_2)}{\tilde{a}_1\, a_1\, \tilde{a}_2 a_2} \;\leq\; 0,\end{aligned}$$

which completes the proof. ∎

Using Lemma 3.1 we can prove the following proposition.

**Proposition 3.1 (R. C. Li)** *Let $a_1 \leq \ldots \leq a_k$ and $\tilde{a}_1 \leq \ldots \leq \tilde{a}_k$ be real numbers of the same sign, then*

$$\sum_{i=1}^{k} \left[ \frac{|\tilde{a}_i - a_i|}{\sqrt{|\tilde{a}_i\, a_i|}} \right]^2 = \min_{\tau} \sum_{i=1}^{k} \left[ \frac{|\tilde{a}_{\tau(i)} - a_i|}{\sqrt{|\tilde{a}_{\tau(i)}\, a_i|}} \right]^2,$$

*where the minimization is taken over all permutations $\tau$ of $\{1, 2, \ldots, k\}$.*

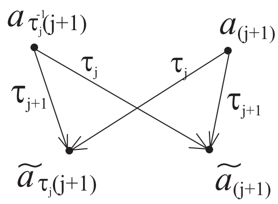


Figure 3.1.1

**Proof**. The proof is similar to the proof of [30, Proposition 2.4]. Let us choose the permutation $\tau$ of $\{1, 2, \ldots, k\}$. The idea of the proof is to construct the sequence of $k+1$ permutations $\tau_j$ such that

$$\tau_0 = \tau, \qquad \tau_k = \text{identity permutation}$$

and for $j = 0, 1, \ldots, k-1$

$$\sum_{i=1}^{k} \left[ \frac{|\tilde{a}_{\tau_j(i)} - a_i|}{\sqrt{|\tilde{a}_{\tau_j(i)}\, a_i|}} \right]^2 \geq \sum_{i=1}^{k} \left[ \frac{|\tilde{a}_{\tau_{j+1}(i)} - a_i|}{\sqrt{|\tilde{a}_{\tau_{j+1}(i)}\, a_i|}} \right]^2 .$$

The construction of these $\tau_j$'s goes as follows: Set $\tau_0 = \tau$. Further, for given $\tau_j$, if $\tau_j(j+1) = j+1$ we set $\tau_{j+1} = \tau_j$; otherwise ($\tau_j(j+1) \neq j+1$) define

$$\tau_{j+1}(i) = \begin{cases} \tau_j(i) & \text{for } \tau_j^{-1}(j+1) \neq i \neq j+1 \\ j+1 & \text{for } i = j+1 \\ \tau_j(j+1) & \text{for } i = \tau_j^{-1}(j+1) \end{cases}$$

Notice that in the latter case $\tau_j(j+1) > j+1$ and $\tau_j^{-1}(j+1) > j+1$. Also notice that $\tau_j$ and $\tau_{j+1}$ differ only in two indices as shown in Figure 3.1.1. Now, Lemma 3.1 implies that

$$\sum_{i=1}^{k} \left[ \frac{|\tilde{a}_{\tau_j(i)} - a_i|}{\sqrt{|\tilde{a}_{\tau_j(i)}\, a_i|}} \right]^2 \geq \sum_{i=1}^{k} \left[ \frac{|\tilde{a}_{\tau_{j+1}(i)} - a_i|}{\sqrt{|\tilde{a}_{\tau_{j+1}(i)}\, a_i|}} \right]^2 .$$

We can continue this construction until the proof is complete. ■

**Remark 3.1** *Lemma 3.1 may fail if not all $a_i's$ and $\tilde{a}_j's$ are of the same sign. Here is a counterexample for $n = 2$: let*

$$a_1 = -1 < a_2 = 0.1 \quad \text{and} \quad \tilde{a}_1 = -0.1 < \tilde{a}_2 = 2.$$

*Then*

$$\frac{(\tilde{a}_1 - a_1)^2}{\tilde{a}_1\, a_1} + \frac{(\tilde{a}_2 - a_2)^2}{\tilde{a}_2\, a_2} = \frac{0.81}{0.1} + \frac{3.61}{0.2} = 26.15 >$$
$$8.5 = \frac{9}{2} + \frac{0.04}{0.01} = \frac{(\tilde{a}_2 - a_1)^2}{\tilde{a}_2\, a_1} + \frac{(\tilde{a}_1 - a_2)^2}{\tilde{a}_1\, a_2}.$$

We also need the following auxiliary lemma on doubly stochastic matrices. We say that a real $n \times n$ matrix $Y$ is doubly stochastic if $Y_{ij} \geq 0$ and $\sum_{k=1}^{n} Y_{ik} = \sum_{k=1}^{n} Y_{ki} = 1$ for $k = 1, 2, \cdots, n$. By Birkhoff's theorem [20, Theorem 8.7.1], a matrix is doubly stochastic if and only if it lies in the convex hull of all permutation matrices. This result has lead to the following lemma [30, Lemma 9.2].

**Lemma 3.2 (R. C. Li)** *Let $Y$ be a $n \times n$ doubly stochastic matrix, and let $M$ be a $n \times n$ complex matrix. Then there exists a permutation $\tau$ of $\{1, 2, \cdots, n\}$ such that*

$$\sum_{i,j=1}^{n} |M_{ij}|^2 Y_{ij} \geq \sum_{i=1}^{n} |M_{i\tau(i)}|^2.$$

**Proof**. For the proof see [28, Lemma 8.2] ∎

Now we are ready to prove our theorem.

**Theorem 3.1** *Let $H$ and $\tilde{H}$ be given as in (3.1). Then*

$$\sqrt{\sum_{i=1}^{n} \left( \frac{|\tilde{\lambda}_i - \lambda_i|}{\sqrt{|\tilde{\lambda}_i||\lambda_i|}} \right)^2} \leq \|V\|\|\tilde{V}\| \frac{\|A^{-1}\|\|\delta A\|_F}{\sqrt{1 - \|A^{-1}\|\|\delta A\|}}, \tag{3.13}$$

*provided that the right-hand side in (3.13) is less than 2.*

**Proof**. From (3.3) it follows that $\delta H = \tilde{H} - H$ can be written as

$$\tilde{H} - H = \tilde{B} J \tilde{B}^* - B J B^* = \tilde{B}(J \widehat{B}^* - \widehat{B}^{-1} J) B^*,$$

Now, by pre- and post- multiplying this equality by $\tilde{U}^*$ from the left and $U$ from the right, we obtain

$$\begin{aligned}
&\tilde{U}^*(\tilde{H}-H)U = \tilde{\Lambda}\tilde{U}^*U - \tilde{U}^*U\Lambda,\\
&\tilde{U}^*\tilde{B}(J\widehat{B}^* - \widehat{B}^{-1}J)B^*U = |\tilde{\Lambda}|^{1/2}\tilde{V}^*(J\widehat{B}^* - \widehat{B}^{-1}J)V|\Lambda|^{1/2}.
\end{aligned}$$

These equalities imply

$$\tilde{\Lambda}\tilde{U}^*U - \tilde{U}^*U\Lambda = |\tilde{\Lambda}|^{1/2}\tilde{V}^*(J\widehat{B}^* - \widehat{B}^{-1}J)V|\Lambda|^{1/2}.$$

Set $S = \tilde{U}^*U$. Then by interpreting the above equality component-wise we have

$$[S]_{ij} = [\tilde{V}^*(J\widehat{B}^* - \widehat{B}^{-1}J)V]_{ij}\frac{\sqrt{|\tilde{\Lambda}_{ii}||\Lambda_{jj}|}}{\tilde{\Lambda}_{ii} - \Lambda_{jj}}, \tag{3.14}$$

for all $i, j \in \{1, \ldots, n\}$. By taking the Frobenius norm we obtain

$$\sum_{i,j=1}^{n}\left(\frac{|\tilde{\lambda}_i - \lambda_j|}{\sqrt{|\tilde{\lambda}_i||\lambda_j|}}\right)^2 |S_{ij}|^2 = \|\tilde{V}^*(J\widehat{B}^* - \widehat{B}^{-1}J)V\|_F^2$$

Since $(|[S]_{ij}|^2)$ is a doubly stochastic matrix, by applying Lemma 3.2 we have

$$\sum_{i,j=1}^{n}\left(\frac{|\tilde{\lambda}_i - \lambda_{\tau(i)}|}{\sqrt{|\tilde{\lambda}_i||\lambda_{\tau(i)}|}}\right)^2 \leq \|\tilde{V}^*(J\widehat{B}^* - \widehat{B}^{-1}J)V\|_F^2, \tag{3.15}$$

for some permutation $\tau$ of $\{1, 2, \ldots, n\}$. Notice that the assumption of the theorem implies that the right-hand side in (3.15) is less then 4. Thus, we have

$$\frac{|\tilde{\lambda}_i - \lambda_{\tau(i)}|}{\sqrt{|\tilde{\lambda}_i||\lambda_{\tau(i)}|}} \leq 2,$$

which can be satisfied only for permutation which does not mix eigenvalues of different signs. Thus, the permutation $\tau$ of $\{1, 2, \ldots, n\}$ satisfies

$$\begin{aligned}
1 \leq \tau(i) \leq l \qquad & 1 \leq i \leq l\\
l+1 \leq \tau(i) \leq n \qquad & l+1 \leq i \leq n.
\end{aligned}$$

Now, Proposition 3.1 implies

$$\begin{aligned}
&\sum_{i=1}^{l}\left(\frac{|\tilde{\lambda}_i-\lambda_i|}{\sqrt{|\tilde{\lambda}_i||\lambda_i|}}\right)^2+\sum_{i=l+1}^{n}\left(\frac{|\tilde{\lambda}_i-\lambda_i|}{\sqrt{|\tilde{\lambda}_i||\lambda_i|}}\right)^2\\
&\leq\sum_{i=1}^{l}\left(\frac{|\tilde{\lambda}_i-\lambda_{\tau(i)}|}{\sqrt{|\tilde{\lambda}_i||\lambda_{\tau(i)}|}}\right)^2+\sum_{i=l+1}^{n}\left(\frac{|\tilde{\lambda}_i-\lambda_{\tau(i)}|}{\sqrt{|\tilde{\lambda}_i||\lambda_{\tau(i)}|}}\right)^2.
\end{aligned}\tag{3.16}$$

Further, the right-hand side in (3.15) can be bounded by

$$\begin{aligned}
\|\tilde{V}^*(J\widehat{B}^*-\widehat{B}^{-1}J)V\|_F &\leq \|V\|\,\|\tilde{V}^*\|\,\|\widehat{B}^{-1}(\widehat{B}J\widehat{B}^*-J)\|_F\\
&\leq \|V\|\,\|\tilde{V}^*\|\,\||\widehat{\Delta}|^{-1/2}\|\||\Delta|^{-1/2}Q^*\delta AQ|\Delta|^{-1/2}\|_F\\
&\leq \|V\|\,\|\tilde{V}^*\|\,\frac{\|A^{-1}\|\|\delta A\|_F}{\sqrt{1-\|A^{-1}\|\|\delta A\|}}.
\end{aligned}$$

Now the proof follows from this and (3.16). ■

By using Lemma 2.1 we can bound $\|\tilde{V}^*\|$ in the terms of $\|V\|$. From (3.10) and (3.8) follows $\tilde{B}=B\widehat{B}$ and $\widehat{B}=I+T_1$. Here the matrix $T_1$ plays the same role as $E_R$ in Lemma 2.1. Note that, the first inequality in (3.9) can be written as

$$\|T_1\| \quad < \quad \frac{1}{2}\cdot\frac{\eta}{\sqrt{1-\eta}},$$

where $\eta$ is defined by (3.6). If we write $\varrho=\frac{1}{2}\frac{\eta}{\sqrt{1-\eta}}$ and if

$$\|V\|^2\varrho<\frac{1}{4},\tag{3.17}$$

then we have the following bound

$$\|\tilde{V}\|\leq\frac{\|V\|}{\sqrt{1-4\cdot\|V\|^2\varrho}}.\tag{3.18}$$

The following result is a simple corollary of the Theorem 3.1. In this corollary we present a perturbation bound which contains only the unperturbed $J$-unitary matrix.

**Corollary 3.1** *Let $H$ and $\tilde{H}$ be given as in Theorem 3.1, If (3.17) holds we have the following bound:*

$$\sqrt{\sum_{i=1}^{n}\left(\frac{|\tilde{\lambda}_i-\lambda_i|}{\sqrt{|\tilde{\lambda}_i||\lambda_i|}}\right)^2} \leq \frac{\|V\|^2}{\sqrt{1-4\cdot\|V\|^2\varrho}}\cdot\frac{\|A^{-1}\|\|\delta A\|_F}{\sqrt{1-\|A^{-1}\|\|\delta A\|}}, \tag{3.19}$$

*provided that right-hand side is less than 2.*

**Proof**. The bound (3.19) is obtained by inserting (3.18) into (3.13). ■

In the positive definite case, that is, for $J=I$, the bounds (3.13) and (3.19) reduce to the result of Theorem 1.3 ([28, Theorem 3.2. pp. 962-963]). We see that the bound (3.19) for indefinite matrices depends on a additional condition number of order $\|V\|^2$. Thus, it is of interest to say more about the magnitude of this condition number.

As we already mentioned in section 2.3.2, Slapničar and Veselić [42] presented a bound for $\|V\|^2$ of the following form: let $B$ be scaled as $B=A^{1/2}D$ then we have the following bound for $\|V\|$

$$\kappa(V)=\|V\|^2\leq\min_{\bar{D}}\sqrt{\kappa(\bar{D}^*\bar{H}\bar{D})},$$

where $\bar{H}=B^*B$ and $\bar{D}J=J\bar{D}$. If $\bar{D}$ is diagonal so that $A_{ii}=1$, then

$$\|V\|^2\leq\sqrt{\kappa(A)}.$$

Another bound for $\|V\|^2$ can be given in terms of $A$ and $\widehat{A}\equiv D^{-*}|H|D^{-1}$. Here $|H|=\sqrt{H^2}=U|\Lambda|U^*$ is the spectral absolute value of $H$, that is, $|H|$ is the positive definite polar factor of $H$. Indeed, by definition, $V$ diagonalizes the pair $(B^*B,J)$, where $H=BJB^*$. That is, $V^*B^*BV=|\Lambda|$, where $\Lambda$ is the eigenvalue matrix of $H$. Then, the eigenvalue decomposition of $H$ is given by $H=U\Lambda U^*$, where $U=BV|\Lambda|^{-1/2}$. Therefore

$$|H|=U|\Lambda|U^*=BVV^*B^*. \tag{3.20}$$

We now have $\|V\|^2=\|VV^*\|=\|B^{-1}|H|B^{-*}\|$. If we set $\widehat{A}=D^{-*}|H|D^{-1}$, then (3.20) implies

$$\|V\|^2=\||\Delta|^{-1/2}Q^*D^{-*}D^*\widehat{A}DD^{-1}Q|\Delta|^{-1/2}\|\leq\|A^{-1}\|\,\|\widehat{A}\|. \tag{3.21}$$

**Remark 3.2** *Since $\|V\|=\|V^{-1}\|$ we also have $\|V\|^2\leq\|A\|\,\|\widehat{A}^{-1}\|$. This and (3.21) implies $\|V\|^4\leq\kappa(A)\kappa(\widehat{A})$. From this we conclude that the factor $\|V\|^2$ from Corollary*

*3.1 will be small when the matrices $A$ and $\widehat{A}$ are well conditioned. An example of such matrices are the scaled diagonally dominant matrices [1]. These matrices have the form $H = D(J+\Psi)D$, where $D$ is diagonal positive definite, $J = \mathrm{diag}\,(\pm 1)$ and $\|\Psi\| < 1$. For these matrices $A = J + \Psi$ and $\|A\|\,\|\widehat{A}^{-1}\| \leq n(1+\|\Psi\|)/(1-\|\Psi\|)$ [55, Theorem 2.29]. Thus we have*

$$\|V\|^2 \;\leq\; n\,\frac{1+\|\Psi\|}{1-\|\Psi\|}, \tag{3.22}$$

Note that bound (3.22) depends on dimension $n$, which can be a drawback. Thus in the following remark we present a bound for $\|V\|$ when $H = D(J+\Psi)D$ is a scaled diagonally dominant matrix, which does not depend explicity on dimension $n$.

**Remark 3.3** *Let $H = D(J+\Psi)D$ be the scaled diagonally dominant matrix, where $D$ is diagonal positive definite, $J = \mathrm{diag}\,(\pm 1)$ and $\|\Psi\| < 1$. We can write $H$ as*

$$H = D(I+\Psi J)^{1/2} J \left[(I+\Psi J)^{1/2}\right]^* D = \tilde{B} J \tilde{B}^*,$$

*where $\tilde{B} = D(I+\Psi J)^{1/2}$. Similarly as in (3.10) we can write $\tilde{B} = D\widehat{B}$ where $\widehat{B} \equiv I + E_R = (I+\Psi J)^{1/2}$. From (3.9) it follows that*

$$\|E_R\| < \frac{1}{2}\,\frac{\|\Psi\|}{\sqrt{1-\|\Psi\|}}.$$

*If $\|\Psi\|_F < 2(\sqrt{26}-1)/25 \approx 0.32$, then $\|E_R\|_F < 1/5$ and*

$$\gamma = \|E_R\|_F/(1-\|E_R\|) < \frac{1}{4}.$$

*Thus, from Lemma 2.1 it follows*

$$\|V\|^2 \;\leq\; \frac{1}{1-4\gamma}. \tag{3.23}$$

*Note that the bound (3.23) does not explicitly depend on $n$ as the bound (3.22), but here we need the stronger assumption on $\|\Psi\|_F$.*

### 3.1.2 Multiplicative perturbations

In the case of multiplicative perturbations, when $H$ is perturbed to $\tilde{H} = (I+E)^*H(I+E)$, we show that our bound does not make difference between positive and indefinite case, and depends only on the magnitude of $\|E\|$.

Our second result is a perturbation bound for eigenvalues of the Hermitian matrix $H$ under the multiplicative perturbation $\tilde{H} = (I+E)^*H(I+E)$. Before we state the next theorem we need one more result given by the following lemma.

**Lemma 3.3 (Elsner and Friedland)** *For any square matrix $M$ there exist a doubly stochastic matrix $Y$ of the same dimension such that $\sigma^2_{\min}(M)Y_{ij} \leq |M_{ij}|^2$.*

**Proof.** For the proof see [18].

Now, we can state the following theorem.

**Theorem 3.2** *Let $H$ be non-singular Hermitian matrix of order $n$, perturbed to $\tilde{H} = (I+E)^*H(I+E)$. If $\|E\|_F < \sqrt{3}/(1+\sqrt{3})$ then*

$$\sqrt{\sum_{i=1}^{n}\left(\frac{|\tilde{\lambda}_i - \lambda_i|}{\sqrt{|\tilde{\lambda}_i||\lambda_i|}}\right)^2} \leq \psi_M\left(\psi_M + \sqrt{1+\psi_M}\right), \tag{3.24}$$

*where*

$$\psi_M = \frac{2\cdot\|E\|_F - \|E\|\|E\|_F}{1-\|E\|}. \tag{3.25}$$

**Proof**. Let $H$ be decomposed as $H = GJG^*$, where $G$ is non-singular and $J = \operatorname{diag}(\pm 1)$ is diagonal matrix whose diagonal elements are the signs of eigenvalues of $H$. Thus, we can write

$$H = GJG^* \quad \text{and} \quad \tilde{H} = (I+E)^*GJG^*(I+E) \equiv \tilde{G}J\tilde{G}^*.$$

Further, by using HSVD (Section 2.3.2) of matrices $G$ and $\tilde{G}$, respectively, we have

$$G = U|\Lambda|^{1/2}X^{-1} \ \text{and} \ \tilde{G} = \tilde{U}|\tilde{\Lambda}|^{1/2}\tilde{X}^{-1}. \tag{3.26}$$

Here $U = \begin{bmatrix} u_1, & \ldots, & u_n \end{bmatrix}$ is unitary matrix whose columns are the eigenvectors of $H$, $\Lambda = \operatorname{diag}(\lambda_1,\ldots,\lambda_n)$ is diagonal matrix whose diagonal elements are the eigenvalues of $H$, and $X$ is non-singular $J$-unitary matrix which simultaneously diagonalizes the pair $(G^*G, J)$. Analogous definitions hold for perturbed quantities.

The fact that $\tilde{X}$ simultaneously diagonalizes the pair $(\tilde{G}^*\tilde{G}, J)$, can be written as

$$G^* \left(I + E + E^* + EE^*\right) G\tilde{X} = J\tilde{X}\tilde{\Lambda}.$$

Premultiply this equality by $X^*$, use $G^*GX = JX\Lambda$, and rearrange to get

$$\Lambda X^*J\tilde{X} - X^*J\tilde{X}\tilde{\Lambda} = -X^*G^* \left(E(I+E)^{-*} + E^*\right) \tilde{G}\tilde{X}. \tag{3.27}$$

From (3.26) it follows that there exists an unitary matrix $U$ such that $GX = U^*|\Lambda|^{1/2}$, and similarly for perturbed quantities. Therefore, we can rewrite (3.27) as

$$\Lambda X^*J\tilde{X} - X^*J\tilde{X}\tilde{\Lambda} = |\Lambda|^{1/2}\Psi|\tilde{\Lambda}|^{1/2},$$

where

$$\Psi = U^* \left(E(I+E)^{-*} + E^*\right) \tilde{U}.$$

By using this equality component-wise, squaring each term, and adding them all together, we obtain

$$\sum_{i,j=1}^{n} \left( \frac{\lambda_i - \tilde{\lambda}_j}{\sqrt{|\lambda_i\tilde{\lambda}_j|}} \right)^2 |\,[X^*J\tilde{X}]_{ij}|^2 = \|\Psi\|_F^2.$$

By applying Lemma 3.3 to the matrix $X^*J\tilde{X}$, we have

$$\sum_{i,j=1}^{n} \left( \frac{\lambda_i - \tilde{\lambda}_j}{\sqrt{|\lambda_i\tilde{\lambda}_j|}} \right)^2 Y_{ij} \leq \|\Psi\|_F^2 \|(X^*J\tilde{X})^{-1}\|^2$$

for some doubly stochastic matrix $Y$. Note that, since the matrix $X^*J\tilde{X}$ is $J$-unitary, we have $\|(X^*J\tilde{X})^{-1}\| = \|X^*J\tilde{X}\|$.

Lemma 3.2 further implies

$$\sum_{i=1}^{n} \left( \frac{\lambda_i - \tilde{\lambda}_{\tau(i)}}{\sqrt{|\lambda_i\tilde{\lambda}_{\tau(i)}|}} \right)^2 \leq \|\Psi\|_F^2 \|X^*J\tilde{X}\|^2. \tag{3.28}$$

Now, we first bound the norms on the right hand side. The assumption of the theorem implies that

$$\|\Psi\|_F = \|E(I+E)^{-*} + E^*\|_F \leq \frac{\|E\|_F}{1 - \|E\|} + \|E\|_F = \psi_M, \tag{3.29}$$

where $\psi_M$ is defined by (3.25).

It remains to bound $\|X^* J\tilde{X}\|$. Since, $\tilde{G} = (I+E)^* G$, from (2.53) it follows that

$$\|X^* J\tilde{X}\| \leq \eta + \sqrt{1+\eta}, \tag{3.30}$$

where $\eta = \|E\|_F/(1-\|E\|)$. Now, by using the fact that $\eta \leq \psi_M$, from (3.30) and (3.28) it follows that

$$\sum_{i=1}^{n} \left( \frac{\lambda_i - \tilde{\lambda}_{\tau(i)}}{\sqrt{|\lambda_i \tilde{\lambda}_{\tau(i)}|}} \right)^2 \quad \leq \quad \psi_M^2 \left( \psi_M + \sqrt{1+\psi_M^2} \right)^2. \tag{3.31}$$

Finally, the assumption that $\|E\|_F < \sqrt{3}/(1+\sqrt{3})$ imply that the right-hand side of (3.31) is not larger then 4. This, in turn, implies that the permutation $\tau$ does not mix eigenvalues with different signs. That is, the permutation $\tau$ must be such that the quotients in (3.31) contain only eigenvalues of the same sign, otherwise at least two terms would be greater or equal to 4. By applying Proposition 3.1 separately to quotients which contain only positive and only negative eigenvalues in (3.31), we conclude that the sum is minimal when $\tau$ is identity. Finally, (3.24) follows by using this and (3.31). ∎

Note that the bound (3.24) does not make difference between positive definite and indefinite case, and it depends only on magnitudes of $\|E\|$ and $\|E\|_F$.

## 3.2 Eigenvector perturbation bounds

In this section we consider the perturbations of eigenvectors, that is, invariant subspaces of non-singular indefinite Hermitian matrices under similar perturbations as in the Section 3.1.1. We present relative version of well-known $\sin\theta$ theorem of Davis and Kahan [4] for indefinite Hermitian matrices under additive and multiplicative perturbations.

Perturbation bounds for eigenvectors of simple eigenvalues were given for scaled diagonally dominant matrices in [1], and for positive definite matrices in [8]. The bound for invariant subspace which corresponds to single, possibly multiple, eigenvalue of an indefinite Hermitian matrix was given in [55, Theorem 2.48]. This bound, given in terms of projections which are defined by Dunford integral as in [26, Section II.1.4], is generalized to invariant subspaces which correspond to a set of neighboring eigenvalues in [50]. In [29, Theorems 4.5 and 4.6] two bounds for invariant subspaces of a graded positive semi-definite matrix were given. We generalize the result of Theorem 1.3 [29, Theorem 4.5] and give the bound for $\|\sin\Theta\|_F$ for a graded non-singular indefinite matrix. Our results are, therefore, relative variants of the well-known $\sin\Theta$ theorems, that is, Theorems 1.7 and 1.8 [4, Section 2], [46, Theorem V.3.4]. Our results also generalize the ones from [55, 50] to subspaces which correspond to arbitrary set of eigenvalues. Our results and the related results from [1, 8, 55, 41] and other works, are also useful in estimating the accuracy of highly accurate algorithms for computing eigenvalue decompositions [1, 8, 53, 39].

We present two perturbation bounds. The first bound considers the perturbations of invariant subspaces of Hermitian matrix $H$ given in the graded form $H = D^*AD$ under perturbation $\tilde{H} = D^*\tilde{A}D$. And the second bound considers perturbations of invariant subspaces of Hermitian matrix $H$ under multiplicative perturbation $\tilde{H} = (I+E)^*H(I+E)$.

### 3.2.1 Additive bounds

In this section we analyze perturbations of invariant subspaces of the graded matrices which are defined by (3.1). Before we state our first theorem we need some additional notations. Let $H = D^*AD$ be $n \times n$ non-singular indefinite Hermitian matrix, and let

$$H = U\Lambda U^* = [\, U_1 \quad U_2\,] \begin{bmatrix} \Lambda_1 & 0 \\ 0 & \Lambda_2 \end{bmatrix} \begin{bmatrix} U_1^* \\ U_2^* \end{bmatrix}, \tag{3.32}$$

be its eigenvalue decomposition, where

$$\Lambda_1 = \operatorname{diag}(\lambda_{i_1}, \lambda_{i_2}, \cdots, \lambda_{i_k}), \quad \Lambda_2 = \operatorname{diag}(\lambda_{j_1}, \lambda_{j_2}, \cdots, \lambda_{j_l}),$$

and $k + l = n$. Therefore, $\Lambda_1$ and $\Lambda_2$ need not contain subsequent eigenvalues. We assume that $\Lambda_1$ and $\Lambda_2$ have no common eigenvalues, in which case $U_1$ and $U_2$ both span simple invariant subspaces according to [46, Definition V.1.2.]. We perturb $H$ into $\tilde{H} = D^*(A + \delta A)D$, where $\delta A$ is a Hermitian perturbation. Similarly as above, we write the eigenvalue decomposition of $\tilde{H}$ as

$$\tilde{H} = \tilde{U}\tilde{\Lambda}\tilde{U}^* = [\, \tilde{U}_1 \quad \tilde{U}_2\,] \begin{bmatrix} \tilde{\Lambda}_1 & 0 \\ 0 & \tilde{\Lambda}_2 \end{bmatrix} \begin{bmatrix} \tilde{U}_1^* \\ \tilde{U}_2^* \end{bmatrix}, \tag{3.33}$$

where $\tilde{\Lambda}_1 = \operatorname{diag}(\tilde{\lambda}_{i_1}, \cdots, \tilde{\lambda}_{i_k})$ and $\tilde{\Lambda}_2 = \operatorname{diag}(\tilde{\lambda}_{j_1}, \cdots, \tilde{\lambda}_{j_l})$.

**Theorem 3.3** *Let $H$ and $\tilde{H}$ be given as above, and let $\|A^{-1}\|\|\delta A\| < 1$. Then $H$ and $\tilde{H}$ have the same inertia, thus $\tilde{U}_1$ and $\tilde{U}_2$ from (3.33) span simple invariant subspaces, as well, and the canonical angles between the subspaces spanned by $U_1$ and $\tilde{U}_1$ are bounded by*

$$\|\sin\Theta\|_F \equiv \|\tilde{U}_2^* U_1\|_F \le \frac{\|A^{-1}\|\|\delta A\|_F}{\sqrt{1 - \|A^{-1}\|\|\delta A\|}} \cdot \frac{\|V_1\|\|\tilde{V}_2\|}{\displaystyle\min_{\substack{1\le p\le k\\ 1\le q\le l}} \frac{|\lambda_{i_p} - \tilde{\lambda}_{j_q}|}{\sqrt{|\lambda_{i_p}\tilde{\lambda}_{j_q}|}}}, \tag{3.34}$$

*provided that the above minimum is greater than zero. Here $V_1$ and $\tilde{V}_2$ are two matrices which are defined similarly as in (3.11) and (3.12).*

**Proof**. The proof is similar to the proof of [29, Theorem 4.3]. Moreover, the first half of the proof is similar to the proof of Theorem 3.1, but we will present a hole proof for the sake of completeness. By using (3.32) we can write

$$H = U|\Lambda|^{1/2}J|\Lambda|^{1/2}U^*,$$

where $J$ is diagonal matrix whose diagonal elements are the signs of the eigenvalues of $H$. By Sylvester's law of inertia [20, Theorem 4.5.8] matrices $H$ and $A$ have the same inertia, so we can chose an eigenvalue decomposition of $A$ such that

$$A = Q\Delta Q^* = Q|\Delta|^{1/2}J|\Delta|^{1/2}Q^*$$

with the same $J$. By using this eigenvalue decomposition we have

$$\begin{aligned} H &= D^*AD = BJB^*, \qquad B = D^*Q|\Delta|^{1/2}, \\ \tilde{H} &= D^*(A+\delta A)D \\ &= B(J + |\Delta|^{-1/2}Q^*\delta AQ|\Delta|^{-1/2})B^* \equiv B\widehat{A}B^*. \end{aligned} \tag{3.35}$$

Since by the assumption

$$\||\Delta|^{-1/2}Q^*\delta AQ|\Delta|^{-1/2}\| \leq \|A^{-1}\|\|\delta A\| < 1,$$

we conclude that the matrices $H$, $\tilde{H}$ and $\widehat{A}$ all have the same inertia defined by $J$. Therefore, $\widehat{A}$ has an eigenvalue decomposition given by

$$\widehat{A} = \widehat{Q}\widehat{\Delta}\widehat{Q}^* = \widehat{Q}|\widehat{\Delta}|^{1/2}J|\widehat{\Delta}|^{1/2}\widehat{Q}^* \equiv \widehat{B}J\widehat{B}^*.$$

Further, (3.33) and (3.35) imply

$$\tilde{H} = \tilde{U}|\tilde{\Lambda}|^{1/2}J|\tilde{\Lambda}|^{1/2}\tilde{U}^* = B\widehat{B}J\widehat{B}^*B^* \equiv \tilde{B}J\tilde{B}^*. \tag{3.36}$$

From (3.32) and (3.35) we conclude that $B$ has the hyperbolic singular value decomposition (see Section 2.3.2) given by

$$B = U|\Lambda|^{1/2}JV^*J, \qquad V^*JV = J,$$

where $V$ is a $J$-unitary matrix which also diagonalizes the positive definite pair $(B^*B, J)$. Similarly, from (3.36) we conclude that $\tilde{B}$ has the hyperbolic singular value decomposition given by

$$\tilde{B} = \tilde{U}|\tilde{\Lambda}|^{1/2} J\tilde{V}^* J, \qquad \tilde{V}^* J\tilde{V} = J.$$

Now

$$\begin{aligned}
&\tilde{H} - H = \tilde{B}J\tilde{B}^* - BJB^* = \tilde{B}(J\widehat{B}^* - \widehat{B}^{-1}J)B^*,\\
&\tilde{U}^*(\tilde{H} - H)U = \tilde{\Lambda}\tilde{U}^*U - \tilde{U}^*U\Lambda,\\
&\tilde{U}^*\tilde{B}(J\widehat{B}^* - \widehat{B}^{-1}J)B^*U = |\tilde{\Lambda}|^{1/2}\tilde{V}^*(J\widehat{B}^* - \widehat{B}^{-1}J)V|\Lambda|^{1/2}.
\end{aligned}$$

From the last equality, (3.32) and (3.33) it follows

$$\tilde{\Lambda}_2\tilde{U}_2^*U_1 - \tilde{U}_2^*U_1\Lambda_1 = |\tilde{\Lambda}_2|^{1/2}\tilde{V}_2^*(J\widehat{B}^* - \widehat{B}^{-1}J)V_1|\Lambda_1|^{1/2}.$$

By interpreting this equality component-wise we have

$$[\tilde{U}_2^*U_1]_{qp} = [\tilde{V}_2^*(J\widehat{B}^* - \widehat{B}^{-1}J)V_1]_{qp}\frac{\sqrt{|\tilde{\Lambda}_{2,qq}||\Lambda_{1,pp}|}}{\tilde{\Lambda}_{2,qq} - \Lambda_{1,pp}}, \tag{3.37}$$

for all $p \in \{1, \ldots, k\}$ and $q \in \{1, \ldots, l\}$. The theorem follows by taking the Frobenius norm and by using

$$\begin{aligned}
&\|\tilde{V}_2^*(J\widehat{B}^* - \widehat{B}^{-1}J)V_1\|_F \le \|V_1\|_2\|\tilde{V}_2^*\|_2\|\widehat{B}^{-1}(\widehat{B}J\widehat{B}^* - J)\|_F\\
&\le \|V_1\|_2\|\tilde{V}_2^*\|_2\||\widehat{\Delta}|^{-1/2}\|_2\||\Delta|^{-1/2}Q^*\delta AQ|\Delta|^{-1/2}\|_F\\
&\le \|V_1\|_2\|\tilde{V}_2^*\|_2\frac{\|A^{-1}\|_2\|\delta A\|_F}{\sqrt{1 - \|A^{-1}\|_2\|\delta A\|_2}}.
\end{aligned}$$

■

Similarly as in the last section we can present a bound which contains only unperturbed $J$-unitary matrix. Using the fact that $\tilde{B} = B\widehat{B}$ and $\widehat{B} = I + T_1$, we can apply result from Lemma 2.1 to bound $\|\tilde{V}^*\|_2$ in the terms of $\|V\|_2$.

If we write $\varrho = \dfrac{1}{2}\dfrac{\eta}{\sqrt{1-\eta}}$ and if (3.17) holds, that is,

$$\|V\|^2\varrho < \frac{1}{4}$$

then we have the following bound

$$\|\tilde{V}\| \le \frac{\|V\|}{\sqrt{1 - 4\cdot\|V\|^2\varrho}}. \tag{3.38}$$

The following result is the simple corollary of the Theorem 3.3.

**Corollary 3.2** *Let $H$ and $\tilde{H}$ be given as in Theorem 3.1, if $\varrho < 1/(4\|V\|^2)$, then:*

$$\|\sin\Theta\|_F \leq \frac{1}{\sqrt{1-4\cdot\|V\|^2\varrho}} \cdot \frac{\|V\|^2}{\min\limits_{\substack{1\leq p\leq k\\1\leq q\leq l}} \frac{|\lambda_{i_p}-\tilde{\lambda}_{j_q}|}{\sqrt{|\lambda_{i_p}\tilde{\lambda}_{j_q}|}}} \frac{\|A^{-1}\|\|\delta A\|_F}{\sqrt{1-\|A^{-1}\|\|\delta A\|}} \tag{3.39}$$

*provided that the right-hand side is less than 2.*

**Proof**. The bound (3.39) is obtained by inserting (3.38) in (3.34). ■

Theorem 1.9 is a proper generalization of Theorem 1.3 [30, Theorem 4.3], since for positive definite $H$ the matrices $\tilde{V}$ and $V$ are unitary. The exception is that the latter theorem also holds in the semi-definite case. We cannot accommodate this condition due to the use of the Sylvester's law of inertia in our proof.

The minimum in (3.19) plays the role of the relative gap, although the function $|\lambda - \tilde{\lambda}|/\sqrt{|\lambda\tilde{\lambda}|}$ does not necessarily increase with the distance between $\lambda$ and $\tilde{\lambda}$ if they have different signs. For example, if $\Lambda_1 = \{1\}$ and $\tilde{\Lambda}_2 = \{-1, 0.1\}$, then the minimum is attained between 1 and $-1$ and not between 1 and 0.1.

### 3.2.2 Multiplicative bounds

Similarly as in section 3.1.2, our second result is the perturbation bound for the canonical angles between unperturbed and perturbed invariant subspace of the Hermitian matrix $H$ under multiplicative perturbation $\tilde{H} = (I+E)^*H(I+E)$. Let $U_1$, $U_2$, $\Lambda_1$ and $\Lambda_2$ be defined as in (3.32) and (3.33).

We have the following theorem.

**Theorem 3.4** *Let $H$ be non-singular Hermitian matrix of order $n$, and let $H$ be perturbed into $\tilde{H} = (I+E)^*H(I+E)$. If $\|E\| < 1$, then*

$$\|\sin\Theta\|_F \equiv \|\tilde{U}_2^* U_1\|_F \leq \left(\frac{\|E\|_F}{1-\|E\|} + \|E\|_F\right) \cdot \frac{1}{\min\limits_{\substack{1\leq p\leq l\\1\leq q\leq k}} \frac{|\lambda_{i_p}-\tilde{\lambda}_{j_q}|}{|\lambda_{i_p}|+|\tilde{\lambda}_{j_q}|}}, \tag{3.40}$$

*where $\Theta$ is diagonal matrix with the canonical angles between the subspaces spanned by $U_1$ and $\tilde{U}_1$.*

**Proof**. Let $H$ be decomposed as $H = GJG^*$, where $G$ is non-singular and $J = \mathrm{diag}\,(\pm 1)$ is diagonal matrix with signs of eigenvalues of matrix $H$. Then we can write

$$H = GJG^* \quad \text{and} \quad \tilde{H} = (I+E)^* GJG^*(I+E) \equiv \tilde{G}J\tilde{G}^*.$$

Since,

$$\tilde{U}^*\left(\tilde{H} - H\right)U = \tilde{\Lambda}\tilde{U}^*U - \tilde{U}^*U\Lambda,$$

and also

$$\tilde{U}^*\left(\tilde{H} - H\right)U = \tilde{U}^*\left(\tilde{H}(I+E)^{-1}E + E^*H\right)U,$$

we can write

$$\tilde{\Lambda}\,\tilde{U}^*U - \tilde{U}^*U\Lambda = \tilde{\Lambda}\tilde{U}^*(I+E)^{-1}EU + \tilde{U}^*E^*U\Lambda.$$

Writing this equality block-wise, for block $(2,1)$ we obtain

$$\tilde{\Lambda}_2\,\tilde{U}_2^*U_1 - \tilde{U}_2^*U_1\Lambda_1 = \tilde{\Lambda}_2\tilde{U}_2^*(I+E)^{-1}EU_1 + \tilde{U}_2^*E^*U_1\Lambda_1.$$

Now, using this equality component-wise for all pairs of indices $(p,q)$, where $1 \leq p \leq k$ and $1 \leq q \leq l$, we obtain

$$|(\tilde{U}_2^*U_1)_{pq})| \leq |(\Lambda_2)_{pp}||\tilde{U}_{2(:,p)}^{\;*}(I+E)^{-1}EU_{1(:,q)}| + |\tilde{U}_{2(:,p)}^{\;*}E^*U_{1(:,q)}||(\Lambda_1)_{qq}|.$$

Since, $\tilde{U}_2$ and $U_1$ are matrices with orthonormal columns, the above inequality implies that

$$\|\tilde{U}_2^*U_1\|_F \leq \frac{1}{\displaystyle\min_{\substack{1\leq p\leq l\\1\leq q\leq k}} \frac{|\lambda_{i_p} - \tilde{\lambda}_{j_q}|}{|\lambda_{i_p}| + |\tilde{\lambda}_{j_q}|}} \cdot (\|(I+E)^{-1}E\|_F + \|E\|_F),$$

which, in turn, implies (3.40). ∎

# Chapter 4

# Relative residual bounds for eigenvalues of indefinite Hermitian matrices

In this chapter we present relative residual bound for the eigenvalues of Hermitian matrices. In the first section we present linear residual bounds for relative distance between eigenvalues and in the second section we present quadratic residual bounds.

## 4.1 Linear residual bounds

In this section we present relative residual bounds for the eigenvalues of Hermitian matrices. First we present three linear relative residual bounds for indefinite non-singular Hermitian matrix. Theorem 4.1 is a proper generalization of Theorem 1.6 [12, Theorem 3]. Theorem 4.3 together with Corollary 4.1 gives a proper generalization of [13, Theorem 1.1]. Besides, Theorems 4.2 and 4.4 give relative residual version of Hoffman - Wielandt's theorem. Second we present a relative residual bound for indefinite possible singular Hermitian matrix. Our result is given in Theorem 4.7 which is a proper generalization of [13, Theorem 2.1].

First we introduce some notation.

Let $H \in \mathbf{C}^{n\times n}$ be a Hermitian matrix, $X \in \mathbf{C}^{n\times m}$ be an orthonormal matrix, and

$$M = X^*HX, \qquad R = HX - XM, \qquad \mathcal{X} = \mathcal{R}(X). \tag{4.1}$$

Furthermore, let

$$\lambda_1 \geq \ldots \geq \lambda_n \quad \text{and} \quad \mu_1 \geq \ldots \geq \mu_m, \tag{4.2}$$

be the eigenvalues of $H$ and $M$, respectively.

The eigenvalues of $M$ are sometimes called Ritz values or Rayleigh-Ritz approximations of the eigenvalues of $H$. Ritz values are optimal in the sense that $\|R\|$ is minimized for $M = X^*HX$, that is, ([46, Theorem 1.15.IV] or [37, Theorem 11-4-5]) if we replace $M$ by another matrix $C$ we can only increase the norm of $R$,

$$\|R\| = \|HX - XM\| \leq \|HX - XC\|,$$

for all matrices $C$ of order $m$.

Moreover, one can always find $m$ eigenvalues of $H$ that are within absolute distance $\|R\|$ of the Ritz values ([37, Theorem 11-5-1])

$$\max_{1\leq j\leq m} |\lambda_{\tau(j)} - \mu_j| \leq \|R\|,$$

for some permutation $\tau$. There is a similar residual bound given in the Frobenius norm ([46, Corollary 4.15.IV])

$$\sqrt{\sum_j (\lambda_{\tau(j)} - \mu_j)^2} \leq \|R\|_F.$$

The above bounds measure absolute distance between eigenvalues, thus they belong to classical or absolute perturbation theory.

Now we present some of relative residual error bounds. Drmač [12] has derived a relative residual error bound which is expressed in terms of angles between the relevant subspaces, $\mathcal{X} = \mathcal{R}(X)$, $\mathcal{Y} = \mathcal{R}(HX)$, $\mathcal{Z} = \mathcal{R}(H^{-1}X)$. The main results from [12] are described in Theorems 1.5 and 1.6.

We present relative residual bounds for the eigenvalues of non-singular indefinite Hermitian matrix $H$ under Hermitian perturbation. The following two theorems contain the relative residual bound which is similar to (1.25).

**Theorem 4.1** *Let $H = LJL^*$, where $L$ and $J$ are non-singular and $J$ is diagonal with $\pm 1$ on its diagonal. Let*

$$\mathcal{Y}_L = L^*\mathcal{X}, \qquad \mathcal{Z}_L = L^{-1}\mathcal{X},$$

*and let $\psi$ be the minimal acute principal angle between $\mathcal{Y}_L$ and $\mathcal{Z}_L$. There are at least $m$ eigenvalues $\lambda_{i_k}$, $k = 1, \ldots, m$ of $H$ for which*

$$\frac{|\lambda_{i_k} - \mu_k|}{|\lambda_{i_k}|} \leq \kappa(V)\frac{\sin\psi}{1-\sin\psi} \quad k = 1,\ldots,m, \tag{4.3}$$

*provided that right hand side in (1.25) is less than one. Here $V$ is $J$-unitary matrix which diagonalizes the pair $(L^*L, J)$, that is, $V^*L^*LV = |\Lambda|$ and $V^*JV = J$.*

**Proof.** Let $|H| = LVV^*L^*$, be the spectral absolute value. By Theorem 1.2 we know that if

$$|x^*\delta Hx| \leq \eta x^*|H|x, \quad \forall\ x,\ \ \eta < 1,$$

then

$$1-\eta \leq \frac{\tilde{\lambda}_i}{\lambda_i} \leq 1+\eta.$$

Now from

$$\begin{aligned} |x^*\delta Hx| &= |x^*LL^{-1}\delta HL^{-*}L^*x| \leq \|L^{-1}\delta HL^{-*}\|\ x^*LL^*x \\ &\leq \|L^{-1}\delta HL^{-*}\| \quad x^*LVV^{-1}V^{-*}V^*L^*x \\ &\leq \|L^{-1}\delta HL^{-*}\| \quad \kappa(V)\ x^*|H|x, \end{aligned}$$

follows that there are at least $r$ eigenvalues $\lambda_{i_k}$ of $H$ such that

$$\frac{|\lambda_{i_k}-\mu_k|}{|\lambda_{i_k}|} \leq \kappa(V)\|L^{-1}\delta HL^{-*}\|. \tag{4.4}$$

From [12, Remark 8] follows that

$$\|L^{-1}\delta HL^{-*}\| = \frac{\sin\psi}{1-\sin\psi},$$

which together with (4.4) yields (4.3). ■

The second theorem contains relative residual version of Hoffman - Wielandt's theorem.

**Theorem 4.2** *Let $H = GJG^*$, where $G$ and $J$ are non-singular and $J$ is diagonal with $\pm 1$ on its diagonal. Let matrix $H = D^*AD$ be the graded form of the matrix $H$, and let $\tilde{H} = D^*(A+\delta A)D$ be a matrix for which $\mathcal{X}$ is invariant subspace. Let both matrices $H$ and $\tilde{H}$ satisfy conditions of Theorem 3.1. Let*

$$\mathcal{Y}_G = G^*\mathcal{X} \qquad \mathcal{Z}_G = G^{-1}\mathcal{X},$$

*and let $\psi$ be the minimal acute principal angle between $\mathcal{Y}_G$ and $\mathcal{Z}_G$. Then there are at least $m$ eigenvalues $\lambda_{i_k}$ $k=1,...,m,$ of $H$ for which*

$$\sqrt{\sum_{k=1}^{m}\left(\frac{|\lambda_{i_k}-\mu_k|}{\sqrt{|\lambda_{i_k}||\mu_k|}}\right)^2} \leq \frac{\|V\|\,\|V\|_F}{\sqrt{1-4\varrho\|V\|^2}}\frac{\sin\psi}{\sqrt{(1-2\sin\psi)\,(1-\sin\psi)}}, \tag{4.5}$$

*Matrix $V$ and $\varrho$ are defined as in Theorem 3.1, provided that the right hand side in (4.5) is positive or* $\sin\psi < 1/2$ *and* $\|A^{-1}\|\|\delta A\|_F < 2/(4\|V\|^2+3)$.

**Proof.** Most of the proof is similar to the proof of Theorem 3.1, but for the sake of the complitnes we present the entire proof. Let $A = Q\Delta Q^* = Q|\Delta|^{1/2}J|\Delta|^{1/2}Q^*$ be an eigenvalue decomposition of $A$. Here $J$ is diagonal matrix of signs whose diagonal elements define the inertia of $A$ and, by Sylvester's theorem [20, Theorem 4.5.8], of $H$, as well. Set

$$G = D^*Q|\Delta|^{1/2}, \tag{4.6}$$

such that $H = GJG^*$. Further, set

$$N = |\Delta|^{-1/2}Q^*\delta AQ|\Delta|^{-1/2}, \tag{4.7}$$

such that $\tilde{H} = G(J+N)G^*$.

By [12, Remark 8] one can see that

$$\|G^{-1}\delta HG^{-*}\| = \frac{\sin\psi}{1-\sin\psi}. \tag{4.8}$$

Further, since $\tilde{H} = G(J+N)G^*$ we have

$$G^{-1}\delta HG^{-*} = N, \tag{4.9}$$

where $N$ is defined by (4.7). Note that (4.8) together with $\|N\| < 1$ implies $\sin\psi < 1/2$.

By assumption we have $\|A^{-1}\|\|\delta A\| < 1$ which together with (4.7) gives $\|N\| < 1$. Thus, we conclude that the matrices $H = GJG^*$, $J+N$ and $\tilde{H} = G(J+N)G^*$ all have the same inertia defined by $J$. Therefore, eigenvalue decompositions of $H$ and $\tilde{H}$ can be written as

$$H = U\Lambda JU^*, \qquad \tilde{H} = \tilde{U}\tilde{\Lambda}J\tilde{U}^*, \tag{4.10}$$

with the same $J = \operatorname{diag}(\pm 1)$, respectively. Also, $J+N$ can be decomposed as

$$J+N = (I+NJ)^{1/2}J[(I+NJ)^{1/2}]^*, \tag{4.11}$$

which follows from $(I+NJ)^{1/2} = J[(I+NJ)^{1/2}]^*J$. Thus, we can write $\tilde{H}$ as

$$\tilde{H} = G(I+NJ)^{1/2}J[(I+NJ)^{1/2}]^*G^*. \tag{4.12}$$

From (4.10) and $H = GJG^*$ we conclude that the matrix $G$ has the HSVD given by

$$G = U|\Lambda|^{1/2}JV^*J, \qquad V^*JV = J. \tag{4.13}$$

Similarly, from (4.10) and (4.12) we conclude that the matrix $G(I+JN)^{1/2}$ has the HSVD given by

$$G(I+NJ)^{1/2} = \tilde{U}|\tilde{\Lambda}|^{1/2}J\tilde{V}^*J, \qquad \tilde{V}^*J\tilde{V} = J. \tag{4.14}$$

Now (4.12) and (4.11) imply that

$$\begin{aligned} \tilde{H} - H &= G(I+NJ)^{1/2}J[(I+NJ)^{1/2}]^*G^* - GJG^* \\ &= G(I+NJ)^{1/2}\,\Xi\, G^*, \end{aligned} \tag{4.15}$$

where

$$\Xi = J[(I+NJ)^{1/2}]^* - (I+NJ)^{-1/2}J = (I+NJ)^{-1/2}N. \tag{4.16}$$

Pre- and post-multiplication of (4.15) by $\tilde{U}$ and $U$, respectively, together with eigenvalue decompositions (4.10), and relations (4.13) and (4.14), gives

$$\tilde{\Lambda}\tilde{U}^*U - \tilde{U}^*U\Lambda = |\tilde{\Lambda}|^{1/2}J\tilde{V}^*J\,\Xi\,JVJ|\Lambda|^{1/2}.$$

By interpreting this equality component-wise we have

$$S_{qp} = [J_2\tilde{V}_2^*J\,\Xi\,JV_1J_1]_{qp}\frac{\sqrt{|\tilde{\Lambda}_{2,qq}||\Lambda_{1,pp}|}}{\tilde{\Lambda}_{2,qq} - \Lambda_{1,pp}},$$

where $S = \tilde{U}^*U$, and $p, q \in \{1, \dots, n\}$. By taking the Frobenius norm we have

$$\sum_{p,q=1}^{n}\left(\frac{|\tilde{\lambda}_p - \lambda_q|}{\sqrt{|\tilde{\lambda}_p||\lambda_q|}}\right)^2 |S_{pq}|^2 = \|J\tilde{V}^*J\,\Xi\,JVJ\|_F^2$$

Since $(|S_{pq}|^2)$ is a doubly stochastic matrix, applying Lemma 3.2 we obtain

$$\sum_{p=1}^{n}\left(\frac{|\tilde{\lambda}_p - \lambda_{\tau(p)}|}{\sqrt{|\tilde{\lambda}_p||\lambda_{\tau(p)}|}}\right)^2 \le \|J\tilde{V}^*J\,\Xi\,JVJ\|_F^2 \tag{4.17}$$

for some permutation $\tau$ of $\{1, 2, \dots, n\}$. Further,

$$\|J\tilde{V}^*J\,\Xi\,JVJ\|_F \le \|\tilde{V}\|\,\|V\|\,\|\Xi\|_F.$$

The relations (4.16), (4.7) and the assumption imply

$$\begin{aligned} \|\Xi\|_F &\le \|(I+JN)^{-1/2}\|\,\|N\|_F \le \frac{1}{\sqrt{1-\|N\|}}\|N\|_F \\ &\le \frac{\|A^{-1}\|\,\|\delta A\|_F}{\sqrt{1-\|A^{-1}\|\,\|\delta A\|}}. \end{aligned} \tag{4.18}$$

Now, from (4.17) and (4.18) we obtain

$$\sqrt{\sum_{i,j=1}^{n}\left(\frac{|\lambda_{\tau(i)}-\tilde{\lambda}_j|}{\sqrt{|\lambda_{\tau(i)}||\tilde{\lambda}_j|}}\right)^2} \leq \|V\|_2\|\tilde{V}^*\|_F\frac{\|N\|}{\sqrt{1-\|N\|}}. \tag{4.19}$$

In our case $\lambda_{\tau(i)} \in \{\lambda_1, ..., \lambda_n\}$ and $\tilde{\lambda}_j \in \{\mu_1, ..., \mu_m\}$. From (4.8), (4.9) and (4.19) it follows

$$\sqrt{\sum_{k=1}^{m}\left(\frac{|\lambda_{i_k}-\mu_k|}{\sqrt{|\lambda_{i_k}||\mu_k|}}\right)^2} \leq \frac{\|V\|_2\|\tilde{V}\|_F}{\sqrt{1-\frac{\sin\psi}{1-\sin\psi}}}\frac{\sin\psi}{1-\sin\psi}. \tag{4.20}$$

Finally, the assumption $\|A^{-1}\|\|\delta A\|_F < 2/(4\|V\|^2+3)$, allows us to apply Lemma 2.1 and bound $\|\tilde{V}\|_F$ by $\|V\|_F$ and $\|V\|$, as

$$\|\tilde{V}\|_F \leq \frac{\|V\|_F}{\sqrt{1-4\varrho\|V\|^2}}.$$

Now the proof is completed by inserting this in (4.20). ■

This theorem is a certain generalization of the [12, Theorem 6] to indefinite Hermitian matrices. In positive definite case matrices $V$ and $\tilde{V}$ are orthonormal. Thus, for positive definite case our bound contains only $\sin\psi$ on the right-hand side similar as [12, Theorem 6].

Further, we present relative residual bounds for indefinite, possible singular, Hermitian matrix. First we will separate null subspace as it is shown in [13].

Let $H = GJG^*$ be an indefinite Hermitian matrix, and let $\mathcal{X}$ be $m-$dimensional subspace of $\mathbf{C}^n$. Let $X = \begin{bmatrix} X_1 & X_2 \end{bmatrix}$ and $X_\perp = \begin{bmatrix} X_{\perp,1} & X_{\perp,2} \end{bmatrix}$ are orthonormal bases of $\mathcal{X}$, $\mathcal{X}_\perp$, respectively, such that

$$M = \begin{bmatrix} X_1 & X_2 \end{bmatrix}^* H \begin{bmatrix} X_1 & X_2 \end{bmatrix} = \begin{array}{c} \begin{bmatrix} 0 & \\ & \Lambda_1 \end{bmatrix} \end{array} \begin{array}{l} m-r_M \\ r_M \end{array}, \tag{4.21}$$

$$N = \begin{bmatrix} X_{\perp,1} & X_{\perp,2} \end{bmatrix}^* H \begin{bmatrix} X_{\perp,1} & X_{\perp,2} \end{bmatrix} = \begin{array}{c} \begin{bmatrix} \Lambda_2 & \\ & 0 \end{bmatrix} \end{array} \begin{array}{l} r_N \\ n-m-r_M \end{array}, \tag{4.22}$$

where $r_M = \text{rank }(M)$, $r_N = \text{rank }(N)$.

Now we can write

$$\begin{bmatrix} X & X_\perp \end{bmatrix}^* H \begin{bmatrix} X & X_\perp \end{bmatrix} = \begin{array}{c} \begin{bmatrix} 0 & & \\ & \widehat{H} & \\ & & 0 \end{bmatrix} \end{array} \begin{array}{l} m-r_M \\ r_M+r_N \\ n-m-r_N \end{array},$$

where

$$\widehat{H} = \begin{bmatrix} \Lambda_1 & K^* \\ K & \Lambda_2 \end{bmatrix} \begin{array}{l} r_M \\ r_N \end{array}. \tag{4.23}$$

The following theorem contains relative perturbation bound for the eigenvalues of $H$ and Rayleigh-Ritz approximations of eigenvalues of $H$, that is, eigenvalues of $M$, where $M$ is given by (4.21) .

**Theorem 4.3** *Let $H = GJG^*$, be indefinite Hermitian matrix (possibly singular), where $J$ is diagonal matrix with $\pm 1$ on its diagonal. Let $X$ and $X_\perp$ be orthonormal matrices as in (4.21) and (4.22), if we define $K_S = |\Lambda_2|^{-\frac{1}{2}} K |\Lambda_1|^{-\frac{1}{2}}$ then*

$$\frac{|\lambda_{i_k} - \mu_{m-r_M+k}|}{|\mu_{m-r_M+k}|} \le \|K_S\|, \qquad k = 1, \ldots, r_M, \tag{4.24}$$

$$\mu_k = \lambda_{j_k} = 0, \qquad k = 1, \ldots, m - r_M,$$

$$\frac{|\lambda_{i_k} - \mu_{m+k}|}{|\mu_{m+k}|} \le \|K_S\|, \qquad k = 1, \ldots, r_N, \tag{4.25}$$

$$\mu_k = \lambda_{j_k} = 0, \qquad k = m + r_N + 1, \ldots, n,$$

**Proof.** Let $\widehat{H}_0$ be diagonal matrix

$$\widehat{H}_0 = \begin{bmatrix} \Lambda_1 & 0 \\ 0 & \Lambda_2 \end{bmatrix} \begin{array}{l} r_M \\ r_N \end{array}, \tag{4.26}$$

where $\Lambda_1$ and $\Lambda_2$ are defined by (4.21) and (4.22). Then we can write $\widehat{H}_0 = D^* A D$, where

$$D = \begin{bmatrix} |\Lambda_1|^{1/2} & 0 \\ 0 & |\Lambda_2|^{1/2} \end{bmatrix}, \qquad A = \begin{bmatrix} J_1 & 0 \\ 0 & J_2 \end{bmatrix}.$$

Here $J_1$ and $J_2$ are diagonal matrices with signs of eigenvalues of $\Lambda_1$ and $\Lambda_2$, respectively.

Note that $\widehat{H}$ from (4.23) can be considered as a perturbation of $\widehat{H}_0$. Indeed, since

$$\widehat{H} = D^* D^{-*} \widehat{H} D^{-1} D,$$

we have $\widehat{H} = D^*(A + \delta A)D$, where

$$\delta A = \begin{bmatrix} 0 & K_S^* \\ K_S & 0 \end{bmatrix}.$$

Now Theorem 1.2 and (1.12), (that is, $\eta = \eta_H = \|\delta A\|\|\widehat{A}^{-1}\|$ ) imply that

$$\frac{|\lambda_{i_k} - \mu_{m-r_M+k}|}{|\mu_{m-r_M+k}|} \leq \|\widehat{A}^{-1}\|\|\delta A\|, \qquad k = 1, \ldots, r_M.$$

Since $\|\widehat{A}^{-1}\|\|\delta A\| = \|K_S\|$, we obtained the first part of (4.24). Similarly holds for (4.25). ■

The bound obtained by this theorem is a proper generalization of [13, Theorem 1.1] to indefinite singular Hermitian matrices. Indeed, in the case when $J = I$ that is, for $H = GG^*$ positive semidefinite, $\|K_S\| = \sin\angle\left(\mathcal{Y}_G, \mathcal{U}_G^{\perp}\right)$, where $\mathcal{Y}_G = G^*\mathcal{X}$ , $\mathcal{U}_G = G^*\mathcal{X}^{\perp}$ and angle function is defined by (see [56])

$$\sin\angle\left(\mathcal{Y}_G, \mathcal{U}_G^{\perp}\right) = \min\left\{\|P_{\mathcal{U}_G}P_{\mathcal{Y}_G}\|, \|P_{\mathcal{U}_G^{\perp}}P_{\mathcal{Y}_G^{\perp}}\|\right\},$$

where $P_{\mathcal{M}}$ is orthogonal projector onto $\mathcal{M}$. Since, for the indefinite Hermitian matrix $H = GJG^*$ we can not express $\|K_S\|$ in the terms of sine of canonical angles, in the corollary we present the upper bound for $\|K_S\|$ which contains such a sine.

**Corollary 4.1** *Let $H = GJG^*$, $J$, $X$ and $X_{\perp}$ be as in Theorem 4.3. If $G^*X_1 = 0$ and $G^*X_{\perp,2} = 0$ and if we set $\mathcal{W}_G = \mathcal{R}(JG^*X_{\perp})$ $\mathcal{Y}_G = \mathcal{R}(G^*X)$, then*

$$\|K_S\| \leq \|U\|\|Y\| \sin\phi, \tag{4.27}$$

*where $U = G^*X_{\perp,1}|\Lambda_2|^{-\frac{1}{2}}$, $Y = G^*X_2|\Lambda_1|^{-\frac{1}{2}}$ and $\sin\phi$ is defined by*

$$\sin\phi = \sin\angle\left(\mathcal{Y}_G, \mathcal{W}_G^{\perp}\right) = \min\left\{\|P_{\mathcal{W}_G}P_{\mathcal{Y}_G}\|, \|P_{\mathcal{W}_G^{\perp}}P_{\mathcal{Y}_G^{\perp}}\|\right\}.$$

**Proof.** From $K = X_{\perp,1}HX_2$ and the definition of $K_S$, we have

$$\begin{aligned} K_S &= |\Lambda_2|^{-\frac{1}{2}}K|\Lambda_1|^{-\frac{1}{2}} = |\Lambda_2|^{-\frac{1}{2}}X_{\perp,1}^*HX_2|\Lambda_1|^{-\frac{1}{2}} = |\Lambda_2|^{-\frac{1}{2}}X_{\perp,1}^*GJG^*X_2|\Lambda_1|^{-\frac{1}{2}} \\ &= \left(G^*X_{\perp,1}|\Lambda_2|^{-\frac{1}{2}}\right)^* J\left(G^*X_2|\Lambda_1|^{-\frac{1}{2}}\right) = U^*JY = W^*Y, \end{aligned}$$

where $W = JU$. Note that

$$Y^*JY = |\Lambda_1|^{-\frac{1}{2}}X_2^*GJG^*X_2|\Lambda_1|^{-\frac{1}{2}}$$

$$= \ |\Lambda_1|^{-\frac{1}{2}} X_2^* H X_2 |\Lambda_1|^{-\frac{1}{2}} = |\Lambda_1|^{-\frac{1}{2}} \Lambda_1 |\Lambda_1|^{-\frac{1}{2}} = J_1,$$

$$\begin{aligned} W^* J W &= |\Lambda_2|^{-\frac{1}{2}} X_{\perp,1}^* G J G^* X_{\perp,1} |\Lambda_2|^{-\frac{1}{2}} \\ &= |\Lambda_2|^{-\frac{1}{2}} X_{\perp,1}^* H X_{\perp,1} |\Lambda_2|^{-\frac{1}{2}} = |\Lambda_2|^{-\frac{1}{2}} \Lambda_2 |\Lambda_2|^{-\frac{1}{2}} = J_2. \end{aligned}$$

This shows that $W$ and $Y$ have $J$-orthogonal columns. Further, from

$$\mathcal{R}(Y) = \mathcal{R}\left(G^* X_2 |\Lambda_1|^{-\frac{1}{2}}\right) = \mathcal{R}\left(G^* X_2\right) \subset \mathcal{R}\left(G^* X\right) = \mathcal{Y}_G,$$

$$\mathcal{R}(W) = \mathcal{R}\left(J G^* X_{\perp,1} |\Lambda_2|^{-\frac{1}{2}}\right) = \mathcal{R}\left(J G^* X_{\perp}\right) \subset \mathcal{R}\left(J G^* X\right) = \mathcal{W}_G,$$

and from $G^* X_1 = 0$ and $G^* X_{\perp,2} = 0$ it follows that $\mathcal{R}(Y) = \mathcal{Y}_G$ and $\mathcal{R}(W) = \mathcal{W}_G$. Finally, let $W = Q_W R_W$ and $Y = Q_Y R_Y$ be QR-decompositions of $W$ and $Y$, respectively. Then

$$\|K_S\| = \|W^* Y\| \leq \|R_W^*\| \|R_Y\| \|Q_W^* Q_Y\|. \tag{4.28}$$

The columns of $Q_W$ and $Q_Y$ form orthonormal basis for $\mathcal{W}_G$ and $\mathcal{Y}_G$, respectively. Drmač and Hari in proof of Theorem 1.1 in [13] have shown that $\|Q_W^* Q_Y\| = \sin\phi$. Now, using this and the fact that

$$\|R_W^*\| = \|U\|, \qquad \|R_Y\| = \|Y\|,$$

from (4.28) follows (4.27). ■

Inserting (4.27) into (4.24) and (4.25) we obtain the bound which is a proper generalization of [13, Theorem 1.1] to indefinite Hermitian matrices, since in semidefinite case ($J = I$) this bound is equal to $\sin\phi$, and $W \equiv U$ and $Y$ have orthonormal columns.

Finally, we present one more bound for indefinite possible singular Hermitian matrices, which can be derived from Theorem 4.2.

Let $H = GJG^*$ be an indefinite possible singular Hermitian matrix, where $G \in \mathbf{C}^{n\times r}$ has a full column rank and $J$ is diagonal matrix with $\pm 1$ on its diagonal ($J$ is inertia of non-zero eigenvalues of $H$). Let $\widehat{H}$ be defined by (4.23). Note that the matrix $\widehat{H}$ is non-singular matrix of dimension $r \times r$ (that is, $r = r_M + r_N$), and note that $\widehat{H}$ can be factorized as $\widehat{H} = LJL^*$, where

$$L = \begin{bmatrix} X_2 & X_{\perp,1} \end{bmatrix}^* G. \tag{4.29}$$

Since factor $L$ is a regular matrix, we can apply Theorem 4.2 (more precisely, we can apply (4.20) from the proof of Theorem 4.2) on $\widehat{H}$. Thus, we proceed as in Theorem 4.2. Let

$$W = \begin{array}{cc} \begin{bmatrix} I \\ 0 \end{bmatrix} & \begin{array}{c} r_M \\ r_N \end{array} \end{array} \tag{4.30}$$

be orthonormal basis with similar role as $X$ in (4.1). Then we have

$$M_0 = W^* \widehat{H} W = \Lambda_1, \qquad R = \widehat{H} W - W M_0 = \begin{bmatrix} 0 \\ K \end{bmatrix}. \tag{4.31}$$

Matrix $\widehat{H}_t = H - \delta H$ has $\mathcal{W} = \mathcal{R}(W)$ for invariant subspace, where

$$\delta H = RW^* + WR^* = \begin{bmatrix} 0 & K^* \\ K & 0 \end{bmatrix}. \tag{4.32}$$

If we write $\widehat{H}_t = \tilde{L} J \tilde{L}^*$ then

$$\tilde{L} = \begin{bmatrix} |\Lambda_1|^{1/2} & 0 \\ 0 & |\Lambda_2|^{1/2} \end{bmatrix}. \tag{4.33}$$

Let $\widehat{H} = U \Lambda_{\widehat{H}} U^*$ be eigenvalue decomposition of $\widehat{H} = LJL^*$. Then the hyperbolic singular value decomposition of matrix $L$ has the form

$$L = U |\Lambda_{\widehat{H}}|^{1/2} J V^* J, \tag{4.34}$$

where $V^* J V = J$.

The following theorem contains relative residual bound for eigenvalues of a indefinite singular Hermitian matrix $H$ under Hermitian perturbation.

**Theorem 4.4** *Let $H = GJG^*$, where $G \in \mathbf{C}^{n \times r}$ has a full column rank, and $J$ is diagonal matrix with $\pm 1$ on the diagonal. Let $X$ and $X_\perp$ be orthonormal matrices as in (4.21) and (4.22). Then*

$$\sqrt{\sum_{k=1}^{r_M} \left( \frac{|\lambda_{i_k} - \mu_{m-r_M+k}|}{\sqrt{|\lambda_{i_k}||\mu_{m-r_M+k}|}} \right)^2} \leq \sqrt{r}\, \|V\| \frac{\sin\psi}{\sqrt{(1 - 2\sin\psi)\,(1 - \sin\psi)}}, \tag{4.35}$$

$$\mu_k = \lambda_{j_k} = 0, \qquad k = 1, \ldots, m - r_M.$$

*In (4.35) $\psi$ is the minimal acute principal angle between $\mathcal{Y}_G = \mathcal{R}(G^* X_2)$ and $\mathcal{Z}_G = \mathcal{R}(G^* X_{\perp,1})^\perp$. Further*

$$\sqrt{\sum_{k=1}^{r_N} \left( \frac{|\lambda_{i_k} - \mu_{m+k}|}{\sqrt{|\lambda_{i_k}||\mu_{m+k}|}} \right)^2} \leq \sqrt{r}\, \|V\| \frac{\sin\theta}{\sqrt{(1 - 2\sin\theta)\,(1 - \sin\theta)}}, \tag{4.36}$$

$$\mu_k = \lambda_{j_k} = 0, \qquad k = m + r_N + 1, \ldots, n.$$

*In (4.36) $\theta$ is the minimal acute principal angle between $\mathcal{R}(G^* X_2)^\perp$ and $\mathcal{R}(G^* X_{\perp,1})$. Matrix $V$ in (4.35) and (4.36) is defined by (4.34), and (4.35) and (4.36) hold provided that the right hand sides are positive.*

**Proof.** As mentioned above, we apply bound (4.20) on matrix $\widehat{H}$. Note that $\widehat{H}$ and its perturbation $\widehat{H}_t = H - \delta H$, where $\delta H$ is given by (4.32), can be decomposed as

$$\widehat{H} = LJL^* \quad \text{and} \quad \widehat{H}_t = \tilde{L}J\tilde{L}^*,$$

respectively, where $L$ is given by (4.29) and $\tilde{L}$ by (4.33). Let (4.34) be an HSVD of $L$. Matrix $\tilde{L}$ is diagonal so it does not require HSVD (that is, $\tilde{U} = I$ and $\tilde{V} = I$). Let $W$ be defined by (4.30). Then we have

$$M_0 = W^*\widehat{H}W = \Lambda_1, \qquad R = HW - WM_0 = \begin{bmatrix} 0 \\ K \end{bmatrix}.$$

Let

$$\lambda_1 \geq \ldots \geq \lambda_{r_M}, \qquad \mu_{m-r_M+1} \geq \ldots \geq \mu_m$$

be the eigenvalues of $M_0$ and $\widehat{H}$, respectively.

Now, all of this, and bound (4.20) give

$$\sqrt{\sum_{k=1}^{r_M} \left( \frac{|\lambda_{i_k} - \mu_{m-r_M+1}|}{\sqrt{|\lambda_{i_k}||\mu_{m-r_M+1}|}} \right)^2} \leq \|V\|\|I\|_F \frac{\sin\psi}{\sqrt{(1-2\sin\psi)\,(1-\sin\psi)}}, \tag{4.37}$$

provided that the right hand side in (4.37) is positive or $\sin\psi < 1/2$. Here $\psi$ is minimal acute principal angle between $\mathcal{Y}_L = L^*\mathcal{W}$ and $\mathcal{Z}_L = L^{-1}\mathcal{W}$.

We need to show that $\mathcal{Y}_G = \mathcal{Y}_L$ and $\mathcal{Z}_G = \mathcal{Z}_L$. For this purpose, let $Y_L = L^*W$ and $Z_L = L^{-1}W$, be matrices whose columns spanned $\mathcal{Y}_L$ and $\mathcal{Z}_L$ respectively, that is, $\mathcal{Y}_L = \mathcal{R}(Y_L)$ and $\mathcal{Z}_L = \mathcal{R}(Z_L)$. Note that

$$Y_L = G^* \begin{bmatrix} X_2 & X_{\perp,1} \end{bmatrix} \begin{bmatrix} I \\ 0 \end{bmatrix} = G^* X_2,$$

which means that $\mathcal{Y}_G = \mathcal{Y}_L$. Before we show that $\mathcal{Z}_G = \mathcal{Z}_L$ note that

$$Z_L = L^{-1} \begin{bmatrix} I \\ 0 \end{bmatrix} = \left( L^* \begin{bmatrix} 0 \\ I \end{bmatrix} \right)^{\perp}.$$

Now, since

$$\left( L^* \begin{bmatrix} 0 \\ I \end{bmatrix} \right)^{\perp} = (G^* X_{\perp,1})^{\perp},$$

it follows that $\mathcal{Z}_G = \mathcal{Z}_L$.

Thus, $\psi$ is minimal acute principal angle between $\mathcal{Y}_G$ and $\mathcal{Z}_G$. Now, from (4.37), using the fact that $\|I\|_F = \sqrt{r}$ we obtain (4.35).

Bound (4.36) can be obtained similarly. In this case for $W$ we take

$$W = \begin{array}{c} \begin{bmatrix} 0 \\ I \end{bmatrix} \end{array} \begin{array}{c} r_M \\ r_N \end{array} .$$

■

The bound obtained by our theorem is a certain generalization of [13, Theorem 1.1] to indefinite singular Hermitian matrices. Here the same comments apply as those following Theorem 4.2.

## 4.2 Quadratic residual bounds

In this section we will present quadratic relative residual bound for the eigenvalues of an indefinite singular Hermitian matrix and compare it with results from classical perturbation theory.

The main result of this section is proper generalization of Drmač and Hari's Theorem 2.1 from [13], to indefinite, possible singular Hermitian matrices.

Drmač and Hari [13] proved the quadratic relative residual perturbation bound for semidefinite Hermitian matrix $H$. They also presented an Example which shows that sometimes results from classical perturbation theory can not be applied.

Before we state the theorem, let us briefly present two results which belong to classical perturbation theory.

Let $M$, $R$ and $\lambda_i$ $i = 1, \ldots, n$, $\mu_k$ $k = 1, \ldots, m$ be as in (4.1) and (4.2), and let $\sigma(H)$ denote the spectra of $H$. The first result is due to Sun [49].

**Theorem 4.5 (Sun)** *Let $\mathcal{Y} = \mathcal{R}(Y)$ be an invariant subspace of $H$ with orthonormal basis $Y \in \mathbf{C}^{n\times m}$. Let $\lambda_{j_1} \geq \ldots \geq \lambda_{j_m}$ be the eigenvalues of $Y^*HY$, and $\Lambda_{\mathcal{Y}} = \mathrm{diag}\,(\lambda_{j_1}, \ldots, \lambda_{j_m})$, $\Lambda_{\mathcal{X}} = \mathrm{diag}\,(\mu_1, \ldots, \mu_m)$. If for some $\alpha, \beta \in \mathbf{R}$ and $\delta_0 > 0$, $\sigma(M) \subset [\alpha, \beta]$, $\sigma(H) \setminus \sigma(Y^*HY) \subset (-\infty, \alpha - \delta_0] \cup [\beta + \delta_0, +\infty)$, (or vice versa) and if $\rho \equiv \|R\|_2/\delta_0 < 1$, then for any unitary invariant norm $\|\cdot\|$,*

$$\|\Lambda_{\mathcal{Y}} - \Lambda_{\mathcal{X}}\| \leq \frac{1}{\sqrt{1-\rho^2}} \cdot \frac{\|R\|_2 \|R\|}{\delta_0}.$$

Second result is due to Mathias [33], and this result is generalization of result obtained by Theorem 4.5.

Let

$$H = \begin{bmatrix} A & R_0 \\ R_0^* & B \end{bmatrix} \quad \text{and} \quad \tilde{H} = \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}$$

be Hermitian matrices. For measure of separation between eigenvalues $\lambda_k$, $k = 1, \dots, n$ of matrix $H$ from eigenvalues $\mu_i(B)$ of matrix $B$ we define

$$\delta_k \equiv \min_{i=1,\dots,n} |\lambda_k - \mu_i(B)|.$$

For measure of separation between eigenvalues $\tilde{\lambda}_k$, $k = 1, \dots, n$ of matrix $\tilde{H}$ from eigenvalues $\mu_i(B)$ of matrix $B$ we use

$$\tilde{\delta}_k \equiv \min_{i=1,\dots,n} |\tilde{\lambda}_k - \mu_i(B)|.$$

**Theorem 4.6** *[33, Theorem 1] If $\lambda_k \notin \sigma(B)$, then*

$$|\lambda_k - \tilde{\lambda}_k| \le \delta_k^{-1} \|R_0\|^2,$$

*while if $\tilde{\lambda}_k \notin \sigma(B)$, then*

$$|\lambda_k - \tilde{\lambda}_k| \le \tilde{\delta}_k^{-1} \|R_0\|^2.$$

In the following theorem $\sigma_{\min}(\cdot)$ denotes the smallest singular value of a matrix. We will use the same notation as in Theorem 4.4. For a given nonzero eigenvalue $\lambda$ of $H$ we shall choose the bases $X$ and $X^\perp$ such that

$$\Lambda_1 = \Xi_\lambda \oplus \widehat{\Xi}_\lambda, \qquad \Lambda_2 = \Omega_\lambda \oplus \widehat{\Omega}_\lambda, \tag{4.38}$$

where the diagonals of $\Xi_\lambda$ and $\Omega_\lambda$ approximate $\lambda$ in the sense of Theorem 4.4.

Let $\Lambda_1$ and $\Lambda_2$ be decomposed as

$$\Lambda_1 = \begin{bmatrix} |\Xi_\lambda|^{1/2} & 0 \\ 0 & |\widehat{\Xi}_\lambda|^{1/2} \end{bmatrix} \begin{bmatrix} J_{11} & 0 \\ 0 & J_{22} \end{bmatrix} \begin{bmatrix} |\Xi_\lambda|^{1/2} & 0 \\ 0 & |\widehat{\Xi}_\lambda|^{1/2} \end{bmatrix}, \tag{4.39}$$

$$\Lambda_2 = \begin{bmatrix} |\Omega_\lambda|^{1/2} & 0 \\ 0 & |\widehat{\Omega}_\lambda|^{1/2} \end{bmatrix} \begin{bmatrix} \bar{J}_{11} & 0 \\ 0 & \bar{J}_{22} \end{bmatrix} \begin{bmatrix} |\Omega_\lambda|^{1/2} & 0 \\ 0 & |\widehat{\Omega}_\lambda|^{1/2} \end{bmatrix}, \tag{4.40}$$

where $J_{11}$, $J_{22}$ $\bar{J}_{11}$, $\bar{J}_{22}$ are diagonal matrices with $\pm 1$ on the diagonal. We write $J = J_{11} \oplus J_{22}$, $\bar{J} = \bar{J}_{11} \oplus \bar{J}_{22}$.

**Theorem 4.7** *Let $H$, $\mathcal{X}$ be as in the Theorem 4.4. Let $\lambda > 0$ be an eigenvalue of $H$ of multiplicity $n(\lambda)$ (for $\lambda < 0$ we consider $-H$). Let the orthonormal bases of $\mathcal{X}$ and $\mathcal{X}^\perp$ be chosen such that (4.38) holds. Write $K_S = |\Lambda_2|^{-1/2} K |\Lambda_1|^{-1/2}$, where $K$ is defined by (4.23). Suppose that there exist constants $\alpha > \gamma$ and $\beta > \gamma$ such that*

$$\|\lambda|\Xi_\lambda|^{-1} - J_{11}\| \le \gamma, \qquad \sigma_{\min}(\lambda|\widehat{\Xi}_\lambda|^{-1} - J_{22}) > \alpha, \tag{4.41}$$

$$\|\lambda|\Omega_\lambda|^{-1} - \bar{J}_{11}\| \le \gamma, \qquad \sigma_{\min}(\lambda|\widehat{\Omega}_\lambda|^{-1} - \bar{J}_{22}) > \beta. \tag{4.42}$$

*If $\Xi_\lambda \oplus \Omega_\lambda$ is of order $n(\lambda)$ and $\|K_S\| \le \gamma < 1$, then*

$$\|\lambda \Xi_\lambda^{-1} - I\| \le \frac{1}{1 - \frac{\|K_S\|^2}{\alpha\,\beta}} \frac{\|K_S\|^2}{\beta} \le \frac{\|U\|^2\|Y\|^2}{1 - \frac{\|U\|^2\|Y\|^2 \sin^2\phi}{\alpha\,\beta}} \frac{\sin^2\phi}{\beta}, \tag{4.43}$$

$$\|\lambda \Omega_\lambda^{-1} - I\| \le \frac{1}{1 - \frac{\|K_S\|^2}{\alpha\,\beta}} \frac{\|K_S\|^2}{\alpha} \le \frac{\|U\|^2\|Y\|^2}{1 - \frac{\|U\|^2\|Y\|^2 \sin^2\phi}{\alpha\,\beta}} \frac{\sin^2\phi}{\beta}, \tag{4.44}$$

*where $K_S = U^* J Y$ and $U$, $Y$ and $\sin\phi$ are defined as in Corollary 4.1.*

**Proof.** Our proof is similar to the proof of [13, Theorem 2.1], and most of it can be omitted but we include the whole proof for completeness. Without loss of generality we can assume

$$H = \begin{bmatrix} \Lambda_1 & K^* \\ K & \Lambda_2 \end{bmatrix} \begin{matrix} r_M \\ r_N \end{matrix} . \tag{4.45}$$

where $\Lambda_1$ and $\Lambda_2$ are given by (4.38). Otherwise one can work with $\widehat{H}$ from the proof of the Theorem 4.4. Matrix $H$ is a non-singular matrix of dimension $r \times r$.

Matrices $J$ and $\bar{J}$ are diagonal matrices which contain signs of $\Lambda_1$ and $\Lambda_2$ from (4.38). It is easy to show that under the assumptions (4.41) and (4.42), $J_{11} = I$ and $\bar{J}_{11} = I$, thus $|\Xi_\lambda| = \Xi_\lambda$ and $|\Omega_\lambda| = \Omega_\lambda$. Indeed, let us show that $J_{11} = I$. From (4.41) we have

$$\|\lambda|\Xi_\lambda|^{-1} - J_{11}\| < 1$$

or

$$\max_j \left| \frac{\lambda}{|\lambda_j|} - \operatorname{sign}(\lambda_j) \right| < 1, \quad j = 1, \ldots, \dim(\Xi_\lambda).$$

The last inequality is equivalent to

$$|\lambda - \operatorname{sign}(\lambda_j)|\lambda_j|| < |\lambda_j|,$$

which can not be obtained for $\lambda_j < 0$, thus $\lambda_j > 0$ for all $j$, and we conclude that $J_{11} = I$.

By Sylvester's low of inertia, the matrix

$$H_S(\lambda) = (|\Lambda_1| \oplus |\Lambda_2|)^{-1/2}(H - \lambda I)(|\Lambda_1| \oplus |\Lambda_2|)^{-1/2}$$

has rank $n - n(\lambda)$. It has the following block structure:

$$H_S(\lambda) = \begin{bmatrix} I - \lambda|\Xi_\lambda|^{-1} & 0 & (K_S^{(1,1)})^* & (K_S^{(2,1)})^* \\ 0 & J_{22} - \lambda|\widehat{\Xi}_\lambda|^{-1} & (K_S^{(1,2)})^* & (K_S^{(2,2)})^* \\ (K_S^{(1,1)}) & (K_S^{(1,2)}) & I - \lambda|\Omega_\lambda|^{-1} & 0 \\ (K_S^{(2,1)}) & (K_S^{(2,2)}) & 0 & \bar{J}_{22} - \lambda|\widehat{\Omega}_\lambda|^{-1} \end{bmatrix}.$$

Let $\widehat{H}_S(\lambda)$ be similar matrix to $H_S(\lambda)$ defined by

$$\begin{aligned} \widehat{H}_S(\lambda) &= \Pi^T H_S(\lambda)\Pi \\ &= \begin{bmatrix} I - \lambda|\Xi_\lambda^{-1}| & (K_S^{(1,1)})^* & 0 & (K_S^{(2,1)})^* \\ (K_S^{(1,1)}) & I - \lambda|\Omega_\lambda|^{-1} & (K_S^{(1,2)}) & 0 \\ 0 & (K_S^{(1,2)})^* & J_{22} - \lambda|\widehat{\Xi}_\lambda|^{-1} & (K_S^{(2,2)})^* \\ (K_S^{(2,1)}) & 0 & (K_S^{(2,2)}) & \bar{J}_{22} - \lambda|\widehat{\Omega}_\lambda|^{-1} \end{bmatrix}. \end{aligned}$$

where $\Pi$ denotes an appropriate permutation matrix.

The assumptions (4.41) and (4.42) imply

$$\begin{aligned} \sigma_{\min}((J_{22} - \lambda|\widehat{\Xi}_\lambda|^{-1}) \oplus (\bar{J}_{22} - \lambda|\widehat{\Omega}_\lambda|^{-1})) &\geq \min\{\alpha, \beta\} && (4.46) \\ &> \gamma \geq \|K_S\| \geq \max_{1 \leq i,j \leq 2} \|K_S^{(i,j)}\| && (4.47) \end{aligned}$$

Hence the matrix

$$\begin{aligned} C &= \begin{bmatrix} J_{22} - \lambda|\widehat{\Xi}_\lambda|^{-1} & (K_S^{(2,2)})^* \\ (K_S^{(2,2)}) & \bar{J}_{22} - \lambda|\widehat{\Omega}_\lambda|^{-1} \end{bmatrix} = \begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix}, \\ C_{12} &= C_{21}^* \end{aligned}$$

and its diagonal blocks $C_{11}$ and $C_{22}$ are non-singular. Therefore (see [20, Section 0.7.3])

$$C^{-1} = \begin{bmatrix} \left[C_{11} - C_{12}C_{22}^{-1}C_{21}\right]^{-1} & C_{11}^{-1}C_{12}\left[C_{21}C_{11}^{-1}C_{12} - C_{22}\right]^{-1} \\ \left[C_{21}C_{11}^{-1}C_{12} - C_{22}\right]^{-1} C_{21}C_{11}^{-1} & \left[C_{22} - C_{21}C_{11}^{-1}C_{12}\right]^{-1} \end{bmatrix},$$

provided that all matrices in the brackets are non-singular. However this follows since these matrices are (signed) Schur complements of $C_{11}$ and $C_{22}$ in $C$. By the last assumption $C$ is of order $n - n(\lambda)$ what is also the rank of $\widehat{H}_S(\lambda)$. Since $C$ is non-singular its Schur complement in $\widehat{H}_S(\lambda)$ must be zero ([37, p.183]). Hence

$$\begin{bmatrix} I - \lambda\Xi_\lambda^{-1} & (K_S^{(1,1)})^* \\ (K_S^{(1,1)}) & I - \lambda\Omega_\lambda^{-1} \end{bmatrix} = \begin{bmatrix} 0 & (K_S^{(2,1)})^* \\ (K_S^{(2,1)}) & 0 \end{bmatrix} C^{-1} \begin{bmatrix} 0 & (K_S^{(1,2)})^* \\ (K_S^{(1,2)}) & 0 \end{bmatrix}. \tag{4.48}$$

From (4.48) we obtain

$$I - \lambda\Xi_\lambda^{-1} = (K_S^{(2,1)})^* \left[\bar{J}_{22} - \lambda|\widehat{\Omega}_\lambda|^{-1} - K_S^{(2,2)}(J_{22} - \lambda|\widehat{\Xi}_\lambda|^{-1})^{-1}(K_S^{(2,2)})^*\right]^{-1} K_S^{(2,1)}$$
$$I - \lambda\Omega_\lambda^{-1} = (K_S^{(1,2)}) \left[J_{22} - \lambda|\widehat{\Xi}_\lambda|^{-1} - (K_S^{(2,2)})^*(\bar{J}_{22} - \lambda|\widehat{\Omega}_\lambda|^{-1})^{-1}(K_S^{(2,2)})\right]^{-1} (K_S^{(1,2)})^*.$$

Now applying standard 2-norm to the expressions on the left- and right-hand side we obtain

$$\|I - \lambda\Xi_\lambda^{-1}\| \leq \frac{\|K_S^{(2,1)}\|^2}{\beta - \frac{\|K_S^{(2,2)}\|^2}{\alpha}}$$

$$\|I - \lambda\Omega_\lambda^{-1}\| \leq \frac{\|K_S^{(1,2)}\|^2}{\alpha - \frac{\|K_S^{(2,2)}\|^2}{\beta}}$$

Since

$$\max_{1\leq i,j\leq 2} \|K_S^{(i,j)}\| \leq \|K_S\|$$

the first inequalities of (4.43) and (4.44) are proved. The upper bounds for (4.43) and (4.44) follow from Corollary 4.1. ■

Theorem 4.7 is a proper generalization of [13, Theorem 2.1], since in positive semidefinite case $J = I$ and bounds (4.43) and (4.44) have the same form as the bound from [13, Theorem 2.1].

The following example is indefinite version of Example 2.4 from [13], and it shows that for certain Hermitian matrices results from Theorem 4.5 and Theorem 4.6 can not be applicable.

**Example 4.1** *Let*

$$H = \begin{bmatrix} -10^{10} & 1 & 10^{-13} \\ 1 & -2\cdot 10^{-5} & 10^{-7} \\ 10^{-13} & 10^{-7} & -10^{-5} \end{bmatrix}, \quad X = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \quad X^{\perp} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \\ 1 & 0 \end{bmatrix}.$$

*Then*

$$M = \begin{bmatrix} 2 \cdot 10^{-5} \end{bmatrix}, \quad R = \begin{bmatrix} 1 & 0 & 10^{-7} \end{bmatrix}^*, \quad \|R\| \approx 1.$$

*The separation $\delta$ from Theorem 4.5 is of order $10^{-5}$, and this also holds for separations $\delta_k$ and $\tilde{\delta}_k$ from Theorem 4.6. In Theorem 4.6 $\|R_0\| = \|R\|$. All this means that in this these theorems are not applicable.*

*On the other hand Theorem 4.4 ensure that for some $j_0 \in \{1, 2, 3\}$ holds*

$$\frac{\lambda_{j_0} - 2 \cdot 10^{-5}}{\sqrt{|\lambda_{j_0}| \cdot 2 \cdot 10^{-5}}} < 1.24 \cdot 10^{-2}. \tag{4.49}$$

*Since $\|K_s\| \approx 7.4 \cdot 10^{-3}$, we can take $\gamma = 2 \cdot 10^{-2}$ in Theorem 4.4. If we consider (4.49) and by taking $\beta = 0.9$ we can assume that conditions from (4.41) are satisfied. Now, the bound from Theorem 4.7 yields*

$$\frac{\lambda_{j_0} - 2 \cdot 10^{-5}}{\lambda_{j_0}} < 6.1 \cdot 10^{-5}.$$

# Chapter 5

# Relative perturbation theory for the hyperbolic eigenvalue problem

In this chapter we consider the hyperbolic eigenvalue problem

$$Hx_i = \lambda_i J x_i, \qquad i = 1, 2, \cdots, n, \tag{5.1}$$

where $H$ is a $n \times n$ Hermitian positive-semidefinite matrix, and $J$ is non-singular matrix of the form $J = \mathrm{diag}\,(\pm 1)$. If $H$ is given by $H = G^* G$, where $G$ is $r \times n$ matrix with full-row rank, and $GJG^*$ is non-singular, then hyperbolic singular value decomposition (HSVD) of $G$ (Section 2.3.2) insures that, there always exists a non-singular matrix $X$ such that

$$\begin{aligned} \begin{bmatrix} X_1^* \\ X_0^* \end{bmatrix} H \begin{bmatrix} X_1 & X_0 \end{bmatrix} &= \begin{bmatrix} |\Lambda_r| & \\ & 0_{n-r} \end{bmatrix} \equiv |\Lambda|, \\ \begin{bmatrix} X_1^* \\ X_0^* \end{bmatrix} J \begin{bmatrix} X_1 & X_0 \end{bmatrix} &= \begin{bmatrix} \widehat{J}_1 & \\ & \widehat{J}_2 \end{bmatrix} \equiv \widehat{J}, \end{aligned} \tag{5.2}$$

where $|\Lambda_r| = \mathrm{diag}\,(|\lambda_1|, \ldots, |\lambda_r|)$, and $\widehat{J}$ is some permutation of $J$. The $i$-th eigenvalue of the problem (5.1) is then given by $\lambda_i \equiv |\Lambda_{ii}| \widehat{J}_{ii}$, and the $i$-th column of $X$ is the corresponding eigenvector. We call such eigenvectors hyperbolic, or $J$-unitary, contrary to the standard unitary eigenvectors. As already mentioned in the Section 2.1, the matrix $X$ is called $J$-unitary[1].

The matrix $X$ appears in some other linear algebra problems. For example, $X$ is the eigenvector matrix of the matrix $JH$,

$$X^{-1}(JH)X = JX^* J(JH)X = \Lambda.$$

[1] In [35], the matrix $X$ is also called a hyper-exchange matrix with respect to the signature matrix $J$.

Also, $X$ is the right singular vector matrix of the HSVD of the pair $(G, J)$. The HSVD for the full row-rank $G \in \mathbf{C}^{r\times n}$ has been described in Section 2.3.2 and can be written as

$$G = USF \equiv U \begin{bmatrix} \Sigma_r & 0 \end{bmatrix} \begin{bmatrix} F_1^* \\ F_0^* \end{bmatrix}, \tag{5.3}$$

where $U$ is $r \times r$ unitary matrix and $F^* \equiv \begin{bmatrix} F_1 & F_0 \end{bmatrix}^* = X^{-1}$, that is,

$$U^*U = I, \quad X^*JX = \widehat{J}, \quad \Sigma = \operatorname{diag}(\sigma_i), \quad \sigma_i > 0.$$

Note that (5.3) holds for any $F_0$ which is $J$-orthogonal to $F_1$, that is, for which is $F_0^*JF_1 = 0$.

Such HSVD is used in the highly accurate algorithm for the Hermitian eigenvalue decomposition of a possibly indefinite symmetric (Hermitian) matrix [53, 39]. Further, HSVD is a suitable way to compute the eigenvalue decomposition of the difference of two outer products [57, 35]. The condition number of $X$ appears in other relative perturbation results [51, 54].

Through this section we assume that $\operatorname{rank}(H) = r \leq n$, and we present the relative perturbation bounds for eigenvalues and perturbation bounds for eigenvectors of the problem (5.1) under additive and multiplicative perturbations of $H$. The bounds for eigenvectors will be given in terms of eigenspaces.

Similarly as in Chapter 3 and Section 3.2 we will consider two kinds of perturbations. An additive perturbation is defined as follows: $H$ is given in a graded form, $H = D^*AD$, where $D$ is matrix with full-row rank equal to $r$, and $A$ is positive definite $r \times r$ matrix. The perturbed matrix $\tilde{H}$ is defined by

$$\tilde{H} \equiv H + \delta H = D^*(A + \delta A)D,$$

where we assume that $A + \delta A$ is positive definite as well.

The multiplicative perturbation is defined by a congruence

$$\tilde{H} = H + \delta H = (I + E)^*H(I + E).$$

For both types of perturbation, the perturbed problem shall be denoted by

$$\begin{aligned} \begin{bmatrix} \tilde{X}_1^* \\ \tilde{X}_0^* \end{bmatrix} \tilde{H} \begin{bmatrix} \tilde{X}_1 & \tilde{X}_0 \end{bmatrix} &= \begin{bmatrix} |\tilde{\Lambda}_r| & \\ & 0_{n-r} \end{bmatrix} \equiv |\tilde{\Lambda}|, \\ \begin{bmatrix} \tilde{X}_1^* \\ \tilde{X}_0^* \end{bmatrix} J \begin{bmatrix} \tilde{X}_1 & \tilde{X}_0 \end{bmatrix} &= \begin{bmatrix} \widehat{J}_1 & \\ & \widehat{J}_2 \end{bmatrix} \equiv \widehat{J}, \end{aligned} \tag{5.4}$$

## 5.1 Eigenvalue bounds for additive perturbations

In this section we are going to derive the relative perturbation bounds for eigenvalues and we will show that these bounds are applicable in numerical computations.

Let $H$ be positive definite, and let the perturbation $\delta H$ satisfy $|x^*\delta Hx| \le \eta\, x^*Hx$ for all $x \in \mathbf{C}^n$ and some $\eta < 1$. Then by Theorem 1.2 the pairs $(H, J)$ and $(\tilde{H}, J)$ have the same inertia, and

$$\frac{|\tilde{\lambda}_i - \lambda_i|}{|\tilde{\lambda}_i|} \le \eta. \tag{5.5}$$

Note that $\log_{10}(|\tilde{\lambda}_i - \lambda_i|/|\tilde{\lambda}_i|)$ can be used to measure the number of correct decimal digits in numerical results. For relative additive perturbations, as it has been shown in (1.8) and (1.9) we have the following corollary.

**Corollary 5.1** *Let $H = D^*AD$ and $\tilde{H} = D^*(A + \delta A)D$ be positive definite, where $A$ and $A + \delta A$ are positive definite, as well. Then (5.5) holds with $\eta = \|\delta A\|\,\|A^{-1}\|$, provided that $\eta < 1$.*

**Proof.** The corollary follows from Theorem 1.2. Indeed, from (1.8) follows

$$\eta_H = 0 \quad \text{and} \quad \eta_K \equiv \eta = \|\delta A\|\,\|A^{-1}\|.$$

Now, from (1.9) follows (5.5). ■

Further, Barlow and Slapničar in [2] have considered the following eigenvalue problem:

$$Hx = \lambda Mx, \quad H, M \in \mathbf{C}^{n\times n} \quad x \in \mathbf{C}^n, \tag{5.6}$$

where $H$ and $M$ are Hermitian matrices. The corresponding perturbed problem is given by

$$(H + \Delta H)\tilde{x} = \tilde{\lambda}(M + \Delta M)\tilde{x}, \tag{5.7}$$

where $\Delta H$ and $\Delta M$ are "small" Hermitian perturbations. Let $\lambda_1 \ge \lambda_2 \ge \ldots \ge \lambda_n$ be the eigenvalues of the pencil (5.6) and let $\tilde{\lambda}_1, \ldots, \tilde{\lambda}_n$ be eigenvalues of the perturbed pencil (5.7).

Further, consider the perturbed matrix

$$\tilde{H} \equiv H + \Delta H = H + \delta\, E, \qquad \tilde{M} \equiv M + \Delta M = H + \delta\, F \tag{5.8}$$

where $\delta$ is positive real number, $E = \Delta H/\delta$ and $F = \Delta M/\delta$. Then, let

$$H(\zeta) = H + \zeta E, \qquad M(\zeta) = M + \zeta F, \tag{5.9}$$

where $\zeta \in [0, \delta]$. Now consider the family of generalized eigenproblems

$$H(\zeta)x(\zeta) = \lambda(\zeta)M(\zeta)x(\zeta), \quad \zeta \in [0, \delta]. \tag{5.10}$$

We assume that there exists a nonsingular matrix $X(\zeta) \in \mathbf{C}^{n\times n}$ such that

$$X(\zeta)^*H(\zeta)X(\zeta) = \Omega(\zeta) \qquad X(\zeta)^*M(\zeta)X(\zeta) = J(\zeta), \quad \forall\ \zeta \in [0, \delta],$$

where

$$\Omega(\zeta) = \operatorname{diag}(\omega_1(\zeta), \ldots, \omega_n(\zeta)), \qquad J(\zeta) = \operatorname{diag}(j_1(\zeta), \ldots, j_n(\zeta)),$$

and

$$\omega_i \in \mathbf{R} \qquad j_i \in \{0, 1\}.$$

Let $\mathcal{S}$ be the set of indices given by

$$\mathcal{S} = \{i : H(\zeta)x_i(\zeta) \neq 0, Mx_i(\zeta) \neq 0 \text{ for all } \zeta \in [0, \delta]\}, \tag{5.11}$$

Now, we can state the following theorem.

**Theorem 5.1 (Barlow and Slapničar)** *Let $(\lambda_i(\zeta), x_i(\zeta)$, be the $i$-th eigenpair of the Hermitian pencil in (5.10). Let $S$ be defined by (5.11). If $\lambda_i(\zeta) \in \mathbf{R}$ for all $\zeta \in [0, \delta]$ and $i \in \mathcal{S}$, then*

$$\exp(-\delta\kappa_i) \ \leq \frac{\lambda_i(\delta)}{\lambda_i(0)} \leq \ \exp(\delta\kappa_i), \tag{5.12}$$

*where*

$$\kappa_i(\zeta) \ = \ \max_{\zeta\in[0,\delta]} \left| \frac{x_i^*(\zeta)Ex_i(\zeta)}{x_i^*(\zeta)H(\zeta)x_i(\zeta)} - \frac{x_i^*(\zeta)Fx_i(\zeta)}{x_i^*(\zeta)M(\zeta)x_i(\zeta)} \right|. \tag{5.13}$$

**Proof.** See [2, Corollary 2.3 and Theorem 2.1]. ∎

The main advantage of Theorem 5.1 is that bound (5.12) allow $H$ and $M$ to be singular, and it is local in the sense that each eigenvalue has its own condition number.

We will present a simple corollary of Theorem 5.1 which considers the hyperbolic eigenvalue problem (5.1). Note that (5.12) holds also for reciprocal eigvalues $1/\lambda_i(\zeta)$, which are eigenvalues of the pair $(M(\zeta), H(\zeta))$. Thus, further we will write $\tilde{\lambda}_i = 1/\lambda_i(\delta)$ and $\lambda_i = 1/\lambda_i(0)$.

**Corollary 5.2** *Let $H = D^*AD$, be perturbed to $\tilde{H} = D^*(A + \delta A)D$, where both $A$ and $\tilde{A}$ are $r \times r$ positive definite $D$ has full row rank, and $A^{1/2}DJD^*A^{1/2}$ is non-singular. Then for all nonzero eigenvalues $\lambda_i$ and $\tilde{\lambda}_i$ of the pairs $(H, J)$ and $(\tilde{H}, J)$, respectively, holds*

$$\exp(-\eta_i) \leq \frac{\tilde{\lambda}_i}{\lambda_i} \leq \exp(\eta_i), \tag{5.14}$$

*where*

$$\begin{aligned} \eta_i &\leq \max_{\zeta\in[0,\delta]} \frac{|x_i^*(\zeta)D^*\delta ADx_i(\zeta)|}{|x_i^*(\zeta)D^*A(\zeta)Dx_i(\zeta)|} \\ &\leq \max_{\zeta\in[0,\delta_A]} \|A(\zeta)^{-1/2}\delta AA(\zeta)^{-1/2}\|. \end{aligned} \tag{5.15}$$

*Here $A(\zeta)$ is defined as*

$$A(\zeta) = A + \zeta E_A \quad \zeta \in [0, \delta_A],$$

*where $\delta_A = \|\delta A\|$ and $E_A = \delta A/\delta_A$.*

**Proof.** From the fact that $J$ is not perturbed and from (5.8) follows that $\Delta J = F = 0$. Further from $E = (D^*\delta AD)/\delta$ and (5.13) follows

$$\begin{aligned} \kappa_i &= \frac{1}{\delta} \max_{\zeta\in[0,\delta]} \frac{|x_i^*(\zeta)D^*\delta ADx_i(\zeta)|}{|x_i^*(\zeta)D^*A(\zeta)Dx_i(\zeta)|} \\ &= \frac{1}{\delta} \max_{\zeta\in[0,\delta]} \frac{|x_i^*(\zeta)D^*A(\zeta)^{1/2}A(\zeta)^{-1/2}\delta AA(\zeta)^{-1/2}A(\zeta)^{1/2}Dx_i(\zeta)|}{|x_i^*(\zeta)D^*A(\zeta)^{1/2}A(\zeta)^{1/2}D(\zeta)x_i(\zeta)|}. \end{aligned}$$

Now we can write

$$\eta_i = \max_{\zeta\in[0,\delta]} \frac{|z_i^*(\zeta)A^{-1/2}\delta AA^{-1/2}z_i(\zeta)|}{\|z_i(\zeta)\|^2},$$

where

$$z_i(\zeta) = A(\zeta)^{1/2}Dx_i(\zeta).$$

The bound (5.15) follows from the fact that

$$\max_{\zeta\in[0,\delta]} \frac{|z_i^*(\zeta)A^{-1/2}\delta AA^{-1/2}z_i(\zeta)|}{\|z_i(\zeta)\|^2} \leq \max_{\zeta\in[0,\delta_A]} \|A(\zeta)^{-1/2}\delta AA(\zeta)^{-1/2}\|. \qquad \blacksquare$$

The bound (5.14) is local and the first inequality from (5.15) is also nearly optimal. However, maximum cannot be readily computed. We can compute only the first order approximation for $\delta = 0$. Thus, we need another bound which is applicable and simple to compute. Such a bound is given in the following theorem which gives a Wielandt-Hoffman type bound similarly as in Chapter 3. Here $A$ needs to be non-singular, as well.

**Theorem 5.2** *Let $H = D^*AD$ and $\tilde{H} = D^*(A+\delta A)D$, where $D$ is a full-row rank matrix and $A$ and $A+\delta A$ are positive definite, such that $A^{1/2}DJD^*A^{1/2}$ is non-singular. Let the matrices $X$ and $\tilde{X}$ perform the diagonalizations (5.2) and (5.4), respectively. If $\|A^{-1/2}\delta A A^{-1/2}\|_F < 2/3$, then*

$$\sqrt{\sum_{i=1}^{r}\left(\frac{|\lambda_i - \tilde{\lambda}_i|}{\sqrt{|\lambda_i\tilde{\lambda}_i|}}\right)^2} \leq \psi\left(\frac{1}{2}\psi + \sqrt{1+\frac{1}{4}\psi^2}\right), \tag{5.16}$$

*where*

$$\psi = \frac{\|A^{-1/2}\delta A A^{-1/2}\|_F}{\sqrt{1-\|A^{-1/2}\delta A A^{-1/2}\|}}. \tag{5.17}$$

**Proof.** The relation (5.4) implies

$$(H+\delta H)\begin{bmatrix}\tilde{X}_1 & \tilde{X}_0\end{bmatrix} = J\begin{bmatrix}\tilde{X}_1 & \tilde{X}_0\end{bmatrix}\begin{bmatrix}\tilde{\Lambda}_r & \\ & 0_{n-r}\end{bmatrix}$$

Premultiply this by $X^*$, use (5.2), and rearrange, to get

$$\Lambda X^*J\tilde{X} - X^*J\tilde{X}\tilde{\Lambda} = -X^*\delta H\tilde{X}. \tag{5.18}$$

Now, the $(1,1)$-block on the right-hand side of (5.18) has the form

$$\begin{aligned} X_1^*\delta H\tilde{X}_1 &= X_1^*D^*\delta AD\tilde{X}_1 \\ &= X_1^*D^*A^{1/2}A^{-1/2}\delta AA^{-1/2}(I+A^{-1/2}\delta AA^{-1/2})^{-1/2} \\ &\quad (I+A^{-1/2}\delta AA^{-1/2})^{1/2}A^{1/2}D\tilde{X}_1. \end{aligned} \tag{5.19}$$

Set $G = A^{1/2}D$ such that $H = G^*G$. Note that $G$ also has full-row rank. By combining (5.2) with (5.3), we see that there exists an unitary matrix $U$ such that $GX_1 = U^*|\Lambda_r|^{1/2}$. Similarly, there exists an unitary matrix $\tilde{U}$ such that $\tilde{G}\tilde{X}_1 = \tilde{U}^*|\tilde{\Lambda}_r|^{1/2}$, where

$$\tilde{G} = (I+A^{-1/2}\delta AA^{-1/2})^{1/2}A^{1/2}D,$$

and $\tilde{H} = \tilde{G}^*\tilde{G}$. Therefore, we can rewrite (5.19) as

$$X_1^*\delta H\tilde{X}_1 = |\Lambda_r|^{1/2}\Psi|\tilde{\Lambda}_r|^{1/2},$$

where

$$\Psi = U^*A^{-1/2}\delta AA^{-1/2}(I + A^{-1/2}\delta AA^{-1/2})^{-1/2}\tilde{U}. \tag{5.20}$$

Thus, (5.18) becomes

$$\Lambda_r X_1^* J\tilde{X}_1 - X_1^* J\tilde{X}_1\tilde{\Lambda}_r = -|\Lambda_r|^{1/2}\Psi|\tilde{\Lambda}_r|^{1/2}. \tag{5.21}$$

By using this equality component-wise for all pairs of indices $(i, j)$, squaring each term, and adding them all together, we obtain

$$\sum_{i,j=1}^{r}\left(\frac{\lambda_i - \tilde{\lambda}_j}{\sqrt{|\lambda_i\tilde{\lambda}_j|}}\right)^2 |\,[X_1^* J\tilde{X}_1]_{ij}|^2 = \|\Psi\|_F^2.$$

By applying Lemma 3.3 to the matrix $X^*J\tilde{X}$, we have

$$\sum_{i,j=1}^{r}\left(\frac{\lambda_i - \tilde{\lambda}_j}{\sqrt{|\lambda_i\tilde{\lambda}_j|}}\right)^2 Y_{ij} \le \|\Psi\|_F^2 \cdot \frac{1}{\sigma_{\min}(X_1{}^*J\tilde{X}_1)},$$

for some doubly stochastic matrix $Y$. Note that

$$\frac{1}{\sigma_{\min}(X_1{}^*J\tilde{X}_1)} \le \frac{1}{\sigma_{\min}(X^*J\tilde{X})},$$

and since the matrix $X^*J\tilde{X}$ is $J$-unitary, we have

$$\frac{1}{\sigma_{\min}(X^*J\tilde{X})} = \|(X^*J\tilde{X})^{-1}\| = \|X^*J\tilde{X}\|.$$

Lemma 3.2 further implies

$$\sum_{i=1}^{r}\left(\frac{\lambda_i - \tilde{\lambda}_{\tau(i)}}{\sqrt{|\lambda_i\tilde{\lambda}_{\tau(i)}|}}\right)^2 \le \|\Psi\|_F^2\|X^*J\tilde{X}\|^2, \tag{5.22}$$

for some permutation $\tau$ of $\{1, 2, \cdots, r\}$.

Let us bound the norms on the right hand side. First, the assumption of the theorem implies that

$$\|\Psi\|_F = \|A^{-1/2}\delta A A^{-1/2}(I + A^{-1/2}\delta A A^{-1/2})^{-1/2}\|_F \le \psi, \tag{5.23}$$

where $\psi$ is defined by (5.17).

We will now bound $\|X^* J\tilde{X}\|$. Without loss of generality, we may assume that $J = I_l \oplus (-I_{n-l})$, which can be achieved by suitable permutation. From (2.38) follows that is enough to bound $\|X_-^* J\tilde{X}_+\|$, where the columns of $X_-$ span negative and columns of $\tilde{X}_+$ span positive subspaces of $J$, that is, $\tilde{X}_+^* J\tilde{X}_+ = I_l$ and $X_-^* J X_- = -(I_{n-l})$.

From (5.21) it follows that for all pairs of indices $(i,j)$, where $i = l+1,\cdots,n$ and $j = 1,\cdots,l$, holds

$$(\lambda_i - \tilde{\lambda}_j)\cdot[X^* J\tilde{X}]_{ij} = -\Psi_{ij}|\lambda_i|^{1/2}|\tilde{\lambda}_j|^{1/2}, \tag{5.24}$$

where $\Psi$ is defined by (5.20). Note that in (5.24) some of $\lambda_i$ and $\tilde{\lambda}_j$ can be zero. Moreover, some zero-eigenvalues belong to eigenvectors from $\mathcal{R}(X_-)$ and other belong to eigenvectors from $\mathcal{R}(\tilde{X}_+)$. Thus, let $s$ be the number of zero-eigenvalues which belong to eigenvectors $\tilde{x}_k \in \mathcal{R}(\tilde{X}_+)$ ($\tilde{x}_k^* J\tilde{x}_k = 1,\quad k = 1,\ldots,s$), and let the rest of them belong to eigenvectors $x_k \in \mathcal{R}(X_-)$ ($x_k^* J x_k = -1,\quad k = 1,\ldots,n-r-s$). Note that we can not directly bound all pairs $(i,j)$ in (5.24) since the following situation can occur

$$(0-0)\cdot[X^* J\tilde{X}]_{ij} = 0,$$

where $\lambda_i = 0$ and $\tilde{\lambda}_j = 0$.

Thus, we will need some additional calculations to obtain the bound for the norm of $|[X^* J\tilde{X}]_{ij}|$ for all pairs of $(i,j)$, where $i = l+1,\cdots,n$ and $j = 1,\cdots,l$. We will construct matrices $H_0$ and $\tilde{H}_0$ such that $X^*(H+H_0)X$ and $\tilde{X}^*(\tilde{H}+\tilde{H}_0)\tilde{X}$ are both diagonal and non-singular.

Let the columns of $Q_0$ form the orthonormal basis for $X_0$, that is, $Q_0^* J X_1 = 0$, and similarly for perturbed quantities.

Let $\begin{bmatrix} X_1 & X_0' \end{bmatrix}$ and $\begin{bmatrix} \tilde{X}_1 & \tilde{X}_0' \end{bmatrix}$ perform diagonalizations

$$\begin{aligned}
\begin{bmatrix} X_1^* \\ {X'}_0^* \end{bmatrix}(H + JQ_0Q_0^*J)\begin{bmatrix} X_1 & X'_0 \end{bmatrix} &= \begin{bmatrix} |\Lambda_r| & \\ & |\Omega_{n-r}| \end{bmatrix} \equiv |\Delta| \\
\begin{bmatrix} X_1^* \\ {X'}_0^* \end{bmatrix} J \begin{bmatrix} X_1 & X'_0 \end{bmatrix} &= \begin{bmatrix} \widehat{J}_1 & \\ & \widehat{J}_2 \end{bmatrix},
\end{aligned} \tag{5.25}$$

and

$$\begin{bmatrix} \tilde{X}_1^* \\ \tilde{X}_0^* \end{bmatrix}(\tilde{H} + J\tilde{Q}_0\tilde{Q}_0^*J)\begin{bmatrix} \tilde{X}_1 & \tilde{X}_0 \end{bmatrix} = \begin{bmatrix} |\tilde{\Lambda}_r| & \\ & |\tilde{\Omega}_{n-r}| \end{bmatrix} \equiv |\tilde{\Delta}|$$

$$\begin{bmatrix} \tilde{X}_1^* \\ \tilde{X}_0^* \end{bmatrix} J \begin{bmatrix} \tilde{X}_1 & \tilde{X}'_0 \end{bmatrix} = \begin{bmatrix} \widehat{J}_1 & \\ & \widehat{J}_2 \end{bmatrix}. \tag{5.26}$$

Here $X_0' = X_0 U_0$, and $U_0$ is unitary matrix which diagonalizes $X_0^* J Q_0 Q_0^* J X_0$, that is,

$$U_0^* X_0^* J Q_0 Q_0^* J X_0 U_0 = |\Omega_{n-r}|.$$

Since we have chosen $X'_0$ and $\tilde{X}'_0$ so that $X'^*_0 H X'_0 = 0$ and $\tilde{X}'^*_0 \tilde{H} \tilde{X}'_0 = 0$, it follows that $\begin{bmatrix} X_1 & X'_0 \end{bmatrix}$ and $\begin{bmatrix} \tilde{X}_1 & \tilde{X}'_0 \end{bmatrix}$ also perform diagonalizations (5.2) and (5.4), respectively.

Let $H_0 = J Q_0 Q_0^* J$ and $\tilde{H}_0 = J \tilde{Q}_0 \tilde{Q}_0^* J$, and let $X = \begin{bmatrix} X_+ & X_- \end{bmatrix}$ and $\tilde{X} = \begin{bmatrix} \tilde{X}_+ & \tilde{X}_- \end{bmatrix}$ be a suitable permutation of $\begin{bmatrix} \tilde{X}_1 & \tilde{X}_0' \end{bmatrix}$ and $\begin{bmatrix} X_1 & X_0' \end{bmatrix}$, respectively. That is, this permutation is such that the columns of $X_+$ and $\tilde{X}_+$ span positive subspaces of $J$ and the columns of $X_-$ and $\tilde{X}_-$ span negative subspaces of $J$. Then from (5.25) and (5.26) it follows that $X$ and $\tilde{X}$ simultaneously diagonalize the pairs $(H_{new}, J)$ and $(\tilde{H}_{new}, J)$, respectively, where $H_{new} = H + H_0$ and $\tilde{H}_{new} = \tilde{H} + \tilde{H}_0$, that is,

$$\begin{bmatrix} X_+^* \\ X_-^* \end{bmatrix} H_{new} \begin{bmatrix} X_+ & X_- \end{bmatrix} = \begin{bmatrix} |\Delta_+| & \\ & |\Delta_-| \end{bmatrix}, \qquad \begin{bmatrix} X_+^* \\ X_-^* \end{bmatrix} J \begin{bmatrix} X_+ & X_- \end{bmatrix} = J,$$

$$\begin{bmatrix} \tilde{X}_+^* \\ \tilde{X}_-^* \end{bmatrix} \tilde{H}_{new} \begin{bmatrix} \tilde{X}_+ & \tilde{X}_- \end{bmatrix} = \begin{bmatrix} |\tilde{\Delta}_+| & \\ & |\tilde{\Delta}_-| \end{bmatrix}, \qquad \begin{bmatrix} \tilde{X}_+^* \\ \tilde{X}_-^* \end{bmatrix} J \begin{bmatrix} \tilde{X}_+ & \tilde{X}_- \end{bmatrix} = J.$$

Here $\Delta_+ = \operatorname{diag}(\delta_1, \ldots, \delta_l)$ is diagonal matrix whose diagonal contains all $l - s$ positive eigenvalues $\lambda_i$ of the pair $(H, J)$; the rest of diagonal elements are positive eigenvalues of the pair $(H_{new}, J)$. Similarly, $\Delta_- = \operatorname{diag}(\delta_{l+1}, \ldots, \delta_n)$ is diagonal matrix whose diagonal contains all $r + s - l$ negative eigenvalues $\lambda_i$ of the pair $(H, J)$; the rest of diagonal elements are negative eigenvalues of the pair $(H_{new}, J)$. Similar holds for perturbed quantities.

Now, we can proceed with derivation of the bound for $\|X_-^* J \tilde{X}_+\|$.

First, note that we can write $\tilde{G} = (I + E_L) G$, where

$$I + E_L = (I + A^{-1/2} \delta A A^{-1/2})^{1/2}.$$

Since $\|A^{-1/2} \delta A A^{-1/2}\| < 1$, we can apply Taylor series of the function $(1 + x)^{1/2}$ given by (3.7) to the matrix $(I + A^{-1/2} \delta A A^{-1/2})^{1/2}$ [21, Theorem 6.2.8], which gives

$$\begin{aligned} &(I + A^{-1/2} \delta A A^{-1/2})^{1/2} = \\ &I + \frac{1}{2} A^{-1/2} \delta A A^{-1/2} - \frac{1}{8} (A^{-1/2} \delta A A^{-1/2})^2 + \frac{1}{16} (A^{-1/2} \delta A A^{-1/2})^3 - \ldots, \end{aligned}$$

which implies

$$\|E_L\|_F \leq \frac{1}{2} \cdot \frac{\|A^{-1/2}\delta A A^{-1/2}\|_F}{1 - \|A^{-1/2}\delta A A^{-1/2}\|}.$$

From $JX^{-1} = X\widehat{J}$ follows that we can write

$$G = U^*|\Lambda_r|^{1/2}\widehat{J}_1 X_1^* J, \qquad \tilde{G} = \tilde{U}^*|\tilde{\Lambda}_r|^{1/2}\widehat{J}_1 \tilde{X}_1^* J.$$

Since, $X_1^* J Q_0 = 0$ it follows that $GQ_0 = 0$, which further implies that $\tilde{X}_1^* J Q_0 = 0$. Indeed,

$$\tilde{X}_1^* J \;\; = \;\; \widehat{J}_1|\tilde{\Lambda}_r|^{-1/2}\tilde{U}\tilde{G} = \widehat{J}_1|\tilde{\Lambda}_r|^{-1/2}\tilde{U}(I + E_L)G,$$

thus $GQ_0 = 0$ implies $\tilde{X}_1^* J Q_0 = 0$. Thus, we can take $\tilde{Q}_0 = Q_0$, which further implies that $\delta H_{new} \equiv \tilde{H}_{new} - H_{new} = \delta H$.

Now, as in (5.18) for $H_{new}$ and $\tilde{H}_{new}$ we can write

$$\Delta_- X_-^* J \tilde{X}_+ - X_-^* J \tilde{X}_+ \tilde{\Delta}_+ = -\begin{bmatrix} |\Lambda_-|^{1/2} & \\ & 0 \end{bmatrix} \Psi \begin{bmatrix} |\tilde{\Lambda}_+|^{1/2} & \\ & 0 \end{bmatrix}, \tag{5.27}$$

where $\Lambda_-$ contains all negative eigenvalues of $\Lambda$ and $\tilde{\Lambda}_+$ contains all positive eigenvalues of $\tilde{\Lambda}$, and $\Psi$ is given by (5.20). Thus, for all indices $(i,j)$, where $i = l+1, \cdots, r+s-l$, $j = 1, \cdots, l-s$ from (5.27) follows that

$$|[X^* J \tilde{X}]_{ij}||\delta_i - \tilde{\delta}_j| \leq |\Psi_{ij}|\sqrt{|\delta_i|}\sqrt{|\tilde{\delta}_j|}.$$

For the rest of $(i,j)$, that is, $i = r+s-l+1, \cdots, n$ and $j = l-s+1, \cdots, l$ we have

$$|[X^* J \tilde{X}]_{ij}||\delta_i - \tilde{\delta}_j| \leq 0.$$

Since, for this particular choice of $(i,j)$, where $i = l+1, \cdots, n$ and $j = 1, \cdots, l$, the eigenvalues $\delta_i$ and $\tilde{\delta}_j$ have opposite signs, we can write

$$|\,[X^* J \tilde{X}]_{ij}| \leq \frac{1}{2}|\Psi_{ij}|.$$

Here we have used the fact that $\sqrt{ab}/(a+b) \leq 1/2$ for any two positive numbers $a$ and $b$. By squaring the above inequality and adding all terms together for $i = l+1, \cdots, n$ and $j = 1, \cdots, l$, we obtain

$$\|X_-^* J \tilde{X}_+\|_F \leq \frac{1}{2}\|\Psi\|_F. \tag{5.28}$$

By using this, (2.37) and (5.23), we have

$$\|X^* J\tilde{X}\| \leq \frac{1}{2}\psi + \sqrt{1 + \frac{1}{4}\psi^2}. \tag{5.29}$$

Note that the assumption of the theorem, (5.29) and (5.23) imply that the right hand side of (5.22) is not larger than 4. This, in turn, implies that the permutation $\tau$ must be such that the quotients in (5.22) contain only eigenvalues of the same sign (otherwise at lest two terms would be at least 4). By applying Proposition 3.1 [30, Proposition 2.4] separately to quotients which contain only positive and only negative eigenvalues in (5.22), we conclude that the sum is minimal when $\tau$ is identity. Finally, (5.16) follows by inserting this, (5.23) and (5.29) into (5.22). ■

## 5.2 Eigenvector bounds for additive perturbations

In this section we will present a variant of the well-known $\sin\Theta$ theorems (Theorems 1.7 and 1.8 [4, Section 2], [46, Section V.3]) designed for relative perturbations.

Let $X_1 = \begin{bmatrix} X_{11} & X_{12} \end{bmatrix}$ from (5.2), be partitioned such that

$$\begin{aligned}
\begin{bmatrix} X_{11}{}^* \\ X_{12}^* \end{bmatrix} H \begin{bmatrix} X_{11} & X_{12} \end{bmatrix} &= \begin{bmatrix} |\Lambda_{11}| & \\ & |\Lambda_{22}| \end{bmatrix} \equiv |\Lambda_r|, \\
\begin{bmatrix} X_{11}{}^* \\ X_{12}^* \end{bmatrix} J \begin{bmatrix} X_{11} & X_{12} \end{bmatrix} &= \begin{bmatrix} \widehat{J}_{11} & \\ & \widehat{J}_{22} \end{bmatrix} \equiv \widehat{J}_1.
\end{aligned} \tag{5.30}$$

By [46, Section VI.2.4], columns of $X_{11}$ span a simple eigenspace of the pair $(H, J)$ if $\Lambda_{11}$ and $\Lambda_{22}$ have no common eigenvalues. Similarly, we partitioned $\tilde{X}_1 = \begin{bmatrix} X_{11} & X_{12} \end{bmatrix}$ from (5.4) such that

$$\begin{aligned}
\begin{bmatrix} \tilde{X_{11}}^* \\ \tilde{X}_{12}^* \end{bmatrix} \tilde{H} \begin{bmatrix} \tilde{X}_{11} & \tilde{X}_{12} \end{bmatrix} &= \begin{bmatrix} |\tilde{\Lambda}_{11}| & \\ & |\tilde{\Lambda}_{22}| \end{bmatrix} \equiv |\Lambda_r|, \\
\begin{bmatrix} \tilde{X_{11}}^* \\ \tilde{X}_{12}^* \end{bmatrix} J \begin{bmatrix} \tilde{X}_{11} & \tilde{X}_{12} \end{bmatrix} &= \begin{bmatrix} \widehat{J}_{11} & \\ & \widehat{J}_{22} \end{bmatrix} \equiv \widehat{J}_1.
\end{aligned} \tag{5.31}$$

We set $X = \begin{bmatrix} X_{11} & X_2 \end{bmatrix}$, where $X_2 = \begin{bmatrix} X_{12} & X_0 \end{bmatrix}$. Let

$$V = X^{-1} = \begin{bmatrix} V_{11}^* \\ V_{12}^* \\ V_0^* \end{bmatrix} \equiv \begin{bmatrix} V_{11}^* \\ V_2^* \end{bmatrix}, \tag{5.32}$$

and let

$$\tilde{X}_{11} = \tilde{Q}_{11}\tilde{R}_{11}, \qquad V_2 = Q_2 R_2, \tag{5.33}$$

be the economical QR factorizations of $\tilde{X}_{11}$ and $V_2$, respectively. The columns of $\tilde{Q}_{11}$ and $Q_2$ form the orthogonal bases for $\mathcal{R}(\tilde{X}_{11})$ and $\mathcal{R}(X_{11})^\perp$, respectively. In Section 3.2 we used the fact that if $U\Sigma V^*$ is a singular value decomposition of the matrix $Q_2^*\tilde{Q}_1$, then the diagonal entries of the matrix $\sin\Theta(X_{11}, \tilde{X}_{11}) \equiv \Sigma$ are the sines of canonical angles between the subspaces $\mathcal{R}(X_{11})$ and $\mathcal{R}(\tilde{X}_{11})$ [46, Corollary I.5.4]. Our theorem bounds $\|\sin\Theta(X_{11}, \tilde{X}_{11})\|_F$, and is, therefore, a relative version of $\sin\Theta$ theorems.

Let us define the relative gap between the sets of eigenvalues from $\tilde{\Lambda}_{11}$ and $\Lambda_{22}$ similarly as in Section 3.2,

$$\mathrm{rg}(\tilde{\Lambda}_{11}, \Lambda_{22}) = \min_{i,j} \frac{|\,[\tilde{\Lambda}_{11}]_{ii} - [\Lambda_{22}]_{jj}|}{\sqrt{|\,[\tilde{\Lambda}_{11}]_{ii}\cdot[\Lambda_{22}]_{jj}|}}. \tag{5.34}$$

As we already mentioned in section 3.2 this definition is based on a definition of a relative distance $\mathrm{RelDist}(\lambda, \tilde{\lambda}) = |\lambda - \tilde{\lambda}|/\sqrt{|\lambda\tilde{\lambda}|}$, which was used in [1], [8] and [29, 30]. In the positive definite case (when $J = I$), all $\lambda$ and $\tilde{\lambda}$ are positive and $\mathrm{RelDist}(\lambda, \tilde{\lambda})$ is a generalized metric [29, Proposition 2.4]. In the hyperbolic case, however, $\mathrm{RelDist}(\lambda, \tilde{\lambda})$ does not necessarily increase with the distance between $\lambda$ and $\tilde{\lambda}$ if they have different signs. For example, if $\Lambda_1 = \{1\}$ and $\tilde{\Lambda}_2 = \{-1, 0.1\}$, then the minimum is attained between 1 and $-1$ and not between 1 and 0.1.

Now we can state our theorem.

**Theorem 5.3** *Let $H = D^*AD$ and $\tilde{H} = D^*(A + \delta A)D$, where $D$ is a full-row rank matrix and $A$ and $A + \delta A$ are positive definite, such that $A^{1/2}DJD^*A^{1/2}$ is non-singular. Let the matrices $X$ and $\tilde{X}$ perform the diagonalizations (5.2) and (5.4), respectively. If $\|A^{-1/2}\delta A A^{-1/2}\|_F < 1$, then*

$$\|\sin\Theta(X_1, \tilde{X}_1)\|_F \le \|X\|^2 \left(\frac{1}{2}\psi + \sqrt{1 + \frac{1}{4}\psi^2}\right) \frac{\psi}{\mathrm{rg}(\tilde{\Lambda}_1, \Lambda_2)}, \tag{5.35}$$

*where $\psi$ is defined by (5.17).*

**Proof.** Let (5.32) and (5.33) hold. Then we have

$$\begin{aligned}
\|\sin\Theta(X_{11}, \tilde{X}_{11})\|_F &= \|Q_2^*\tilde{Q}_{11}\|_F = \|R_2^{-*}V_2^*\tilde{X}_{11}\tilde{R}_{11}^{-1}\|_F \\
&\le \|R_2^{-1}\|\,\|\tilde{R}_{11}^{-1}\|\,\|V_2^*\tilde{X}_{11}\|_F \\
&\le \|R_2^{-1}\|\,\|\tilde{R}_{11}^{-1}\|\,\|X_2^*J\tilde{X}_{11}\|_F.
\end{aligned} \tag{5.36}$$

The last inequality follows from (5.32) and the $J$-unitarity of $X$. Using (5.30) we can rewrite (5.2) as

$$\begin{aligned}
\begin{bmatrix} X_{11}{}^* \\ X_{12}^* \\ X_0^* \end{bmatrix} H \,[\, X_{11} \quad X_{12} \quad X_0 \,] &= \begin{bmatrix} |\Lambda_{11}| & & \\ & |\Lambda_{22}| & \\ & & 0 \end{bmatrix}, \\
\begin{bmatrix} X_{11}{}^* \\ X_{12}^* \\ X_0^* \end{bmatrix} J \,[\, X_{11} \quad X_{12} \quad X_0 \,] &= \begin{bmatrix} \widehat{J}_{11} & & \\ & \widehat{J}_{22} & \\ & & \widehat{J}_2 \end{bmatrix}.
\end{aligned} \tag{5.37}$$

Similarly, using (5.31) we can rewrite perturbed quantities from (5.4). Using this, we can interpret the relation (5.18) block-wise, and get

$$\begin{bmatrix} \Lambda_{22} & \\ & 0 \end{bmatrix} \begin{bmatrix} X_{12}^* \\ X_0^* \end{bmatrix} J\tilde{X}_{11} - \begin{bmatrix} X_{12}^* \\ X_0^* \end{bmatrix} J\tilde{X}_{11}\tilde{\Lambda}_{11} = - \begin{bmatrix} |\Lambda_{22}|^{1/2} & \\ & 0 \end{bmatrix} \Psi_{21} |\tilde{\Lambda}_{11}|^{1/2}, \tag{5.38}$$

where

$$\Psi \equiv \begin{bmatrix} \Psi_{11} & \Psi_{12} \\ \Psi_{21} & \Psi_{22} \end{bmatrix}$$

is partitioned according to (5.37), that is, $\Psi_{11}$ has the same dimension as $\Lambda_{11}$, and $\Psi_{22}$ has the same dimension as $\operatorname{diag}(\Lambda_{22}, 0)$. From (5.38) it follows

$$\begin{aligned}
\Lambda_{22} X_{12}^* J\tilde{X}_{11} - X_{12}^* J\tilde{X}_{11}\tilde{\Lambda}_{11} &= -|\Lambda_{22}|^{1/2}\Psi_{21}|\tilde{\Lambda}_{11}|^{1/2}, \\
X_0^* J\tilde{X}_{11} &= 0.
\end{aligned} \tag{5.39}$$

By interpreting (5.39) component-wise and using (5.34), we get

$$|\,[X_{12}^* J\tilde{X}_{11}]_{ji}| \leq \frac{|\,[\Psi_{21}]_{ji}|}{\operatorname{rg}(\tilde{\Lambda}_{11}, \Lambda_{22})},$$

or

$$\|X_{12}^* J\tilde{X}_{11}\|_F \leq \frac{\|\Psi\|_F}{\operatorname{rg}(\tilde{\Lambda}_{11}, \Lambda_{22})} \leq \frac{\psi}{\operatorname{rg}(\tilde{\Lambda}_{11}, \Lambda_{22})}. \tag{5.40}$$

Here the last inequality follows from (5.23). Further,

$$\|R_2^{-1}\| = \frac{1}{\sigma_{\min}(V_2)} \leq \frac{1}{\sigma_{\min}(V)} = \|X\|. \tag{5.41}$$

Similarly,

$$\|\tilde{R}_{11}^{-1}\| \leq \|\tilde{X}^{-1}\| = \|\tilde{X}\|. \tag{5.42}$$

We can further bound $\|\tilde{X}\|$ using (5.29), and the fact that $\|\tilde{X}\| \leq \|X\| \cdot \|X^* J \tilde{X}\|$, which implies

$$\|\tilde{X}\| = \|\tilde{R}_1^{-1}\| \leq \|X\| \left( \frac{1}{2}\psi + \sqrt{1 + \frac{1}{4}\psi^2} \right).$$

Finally, (5.35) follows by inserting this, (5.41) and (5.40) into (5.36). ■

An inconvenience of Theorem 5.3 is that the bound (5.35) uses both original and perturbed eigenvalues in the definition of relative gap. To compute $\mathrm{rg}(\tilde{\Lambda}_1, \Lambda_2)$ one thus needs to know the exact perturbation $\delta A$. However, in some important cases, like the ones described in Section 5.2.1, only information about $\|\delta A\|$ is given. It is therefore desirable to have the bound which uses only original eigenvalues. By using Corollary 5.1, one can easily bound $\mathrm{rg}(\tilde{\Lambda}_1, \Lambda_2)$ from below by

$$\mathrm{rg}(\tilde{\Lambda}_1, \Lambda_2) \geq \min_{i,j} \frac{|\, [\Lambda_1]_{ii}\{1 + \eta \cdot \mathrm{sign}\,([\Lambda_2]_{jj} - [\Lambda_1]_{ii}) \cdot \mathrm{sign}\,([\Lambda_1]_{ii})\} - [\Lambda_2]_{jj} \,|}{\sqrt{|\, [\Lambda_1]_{ii} [\Lambda_2]_{jj} (1 + \eta) \,|}}.$$

Another inconvenience of Theorem 5.3 is that the bound (5.35) contains the term $\|X\|^2$, which is computationally demanding. However, in the important case when the matrix $D$ is diagonal (c.f. Section 5.2.1), we can use (2.35) to bound $\|X\|^2$, thus obtaining the following result.

**Corollary 5.3** *Let $H = DAD$ and $\tilde{H} = D(A + \delta A)D$, where $D$ is diagonal positive definite matrix, and let the matrices $X$ and $\tilde{X}$ perform the diagonalizations (5.30) and (5.31), respectively. If $\|A^{-1/2} \delta A A^{-1/2}\|_F < 2/3$, then*

$$\|\sin \Theta(X_1, \tilde{X}_1)\|_F \leq \sqrt{\kappa(A)} \left( \frac{1}{2}\psi + \sqrt{1 + \frac{1}{4}\psi^2} \right) \frac{\psi}{\mathrm{rg}(\tilde{\Lambda}_1, \Lambda_2)}.$$

### 5.2.1 Applying the bounds

As we have already mentioned in the introduction of this chapter the results of Sections 5.1 and 5.2 enable us to handle two important classes of perturbations which occur in numerical computation. The first class are the perturbations of the form $|\delta H|_{ij} \leq \varepsilon |H_{ij}|$. Such perturbations occur when the matrix $H$ is stored in a computer with machine precision $\varepsilon$. Then, $\|\delta A\| \leq \varepsilon \|\, |A| \,\|$. If we chose $D = \mathrm{diag}\,(\sqrt{|H_{ii}|})$, then further $\|\delta A\| \leq \varepsilon n$. Such choice of $D$ is nearly optimal in the sense that $\kappa(A) \leq n \min_\Delta \kappa(\Delta H \Delta)$, where the minimum is over all non-singular diagonal matrices [43]. The second, more general, class of perturbations has the form $|\delta H|_{ij} \leq \varepsilon\sqrt{|H_{ii}H_{jj}|}$. By choosing the same $D$ as above, we again have $\|\delta A\| \leq \varepsilon n$. Such perturbations occur during various numerical algorithms. Also, for the above perturbations, if $\kappa(A) \ll \kappa(H)$, then our eigenvalue bounds are better than the classical norm-wise bounds like [46, Corollary VI.3.3] which can be applied to the definite matrix pair $(H, J)$. Even more, since the above choice of $D$ is almost optimal, our eigenvalue bounds can never be much worse than the classical bounds. If $\kappa(A) \ll \kappa(H)$, then our eigenvector bound can also be better than the classical bounds like [46, Theorem VI.3.9], in particular when the relative gap is large and the absolute gap is small. This typically occurs in the presence of several tiny eigenvalues.

The following numerical example illustrates some of the above facts.

**Example 5.1** *Let $H = DAD$ with*

$$A = \begin{bmatrix} 1 & 0.8 & 0.8 & 0.8 & 0.8 \\ 0.8 & 1 & 0.8 & 0.8 & 0.8 \\ 0.8 & 0.8 & 1 & 0.8 & 0.8 \\ 0.8 & 0.8 & 0.8 & 1 & 0.8 \\ 0.8 & 0.8 & 0.8 & 0.8 & 1 \end{bmatrix}, \quad D = \begin{bmatrix} 10^{10} & & & & \\ & 50 & & & \\ & & 1 & & \\ & & & 1 & \\ & & & & 10^5 \end{bmatrix},$$

*and let $J = \operatorname{diag}(1, 1, 1, -1, -1)$. The eigenvalues of the pair $(H, J)$ are (properly rounded)*

$$\lambda_1 = 10^{20}, \ \lambda_2 = 722, \ \lambda_3 = 0.254, \ \lambda_4 = -0.254, \ \lambda_5 = -3.6 \cdot 10^7.$$

*Note that $\kappa(H) = 5 \cdot 10^{20}$ while $\kappa(A) = 21$. Let the perturbed matrix $\tilde{H}$ be given as in Theorem 5.2 with*

$$\delta A = 10^{-6} \cdot \begin{bmatrix} 0.46 & -0.26 & -0.02 & 0.36 & 0.42 \\ -0.26 & -0.96 & -0.66 & 0.21 & 0.69 \\ -0.02 & -0.66 & -0.91 & -0.15 & 0.14 \\ 0.36 & 0.21 & -0.15 & 0.27 & -0.33 \\ 0.42 & 0.69 & 0.14 & -0.33 & -0.35 \end{bmatrix}.$$

*This is relative component-wise perturbation satisfying $|\delta A| \leq 10^{-6}|A|$. Also, $\|\delta A\| \leq 2 \cdot 10^{-6}$, $\|\delta A\|_F \leq 2.3 \cdot 10^{-6}$. When applying Theorem 5.2, the left hand side of (5.16) (a norm of the exact perturbation) is equal to $8.4 \cdot 10^{-6}$ and the bound on the right hand side is $1.4 \cdot 10^{-5}$ (with $\psi < 1.4 \cdot 10^{-5}$). Note that the right hand side of (5.16) remains the same for any diagonal grading matrix $D$, thus the bound of Theorem 5.2 is independent of grading. Also, note that we do not need to know the perturbation $\delta A$ exactly, since the bound remains approximately the same for any matrix $\delta A$ which satisfies $|\delta A| \leq 10^{-6}|A|$*

*or* $\|\delta A\| \leq 10^{-6}$. *On the other hand, the bounds of [46, Corollary VI.3.3] are completely useless, since there* $\|E\| = 0$ *but* $\|F\| = \|\delta H\| = 4.6 \cdot 10^{13}$ *and* $\gamma(A, B) = 0.97$ *according to [46, Definition VI.1.17].*

*Let us now consider invariant subspaces. The bound of Corollary 5.3 is obtained by simply multiplying the eigenvalue bound by* $\sqrt{\kappa(A)} = 4.59$ *and dividing it by relative gap. Let* $\sin \Theta_i = \sin \Theta(x_i, \tilde{x}_i)$, *where* $x_i$ *is the eigenvector of* $\lambda_i$. *Table 5.1 gives the corresponding relative gaps, the exact perturbations and the bounds computed by Corollary 5.3.*

| $i$ | *1* | *2* | *3* | *4* | *5* |
|---|---|---|---|---|---|
| $\mathrm{rg}(\tilde{\lambda}_i, \Lambda_2)$ | $1.6 \cdot 10^6$ | 53 | 2 | 2 | 223 |
| $\|\sin \Theta_i\|_F$ | $5 \cdot 10^{-14}$ | $5 \cdot 10^{-8}$ | $10^{-7}$ | $10^{-7}$ | $2 \cdot 10^{-8}$ |
| *Corollary 5.3* | $3 \cdot 10^{-11}$ | $9 \cdot 10^{-7}$ | $3 \cdot 10^{-5}$ | $3 \cdot 10^{-5}$ | $2 \cdot 10^{-7}$ |

Table 5.1: Eigenvector perturbations

The result of Corollary 5.3 can be readily used in the error analysis of some numerical algorithms, in particular the $J$-orthogonal Jacobi method for solving the problem (5.1) from [53] and [39, Section 3.1]. One step of this method in floating-point arithmetic with machine precision $\varepsilon$ induces the backward error $\delta A$ such that $\|\delta A\| \leq \mathrm{const} \cdot \varepsilon$ [39, Theorem 3.2.1]. Theorem 5.2 and Corollary 5.3 then give bounds for errors in eigenvalues and invariant subspaces after one step of the method. By adding these errors together, similarly as in [39, Corollary 3.2.2], we obtain overall error bounds for the method.

## 5.3 Bounds for multiplicative perturbations

In this section we shall give a relative Hoffman-Wielandt type bound, similar to the one of Theorem 5.2, for the case of multiplicative perturbation of a matrix $H$, where $H$ is given as $H = G^* G$, and $G$ is $r \times n$ matrix with full-row rank, assuming also that $GJG^*$ is non-singular. We shall also prove a $\sin \Theta$ type bound, similar to the one of Theorem 5.3.

Multiplicative perturbation of $H$ is given by

$$\tilde{H} = H + \delta H = (I + E)^* H (I + E), \tag{5.43}$$

where, $\|E\| < 1$ so that $(I + E)$ and $G(I + E)J(I + E)^* G^*$ are non-singular.

First note that for multiplicative perturbations, Theorem 5.1 has no such a nice and easy corollary as in the case of additive perturbations. Indeed, let $\tilde{H}$ be as in (5.43) and $M = J$, $\Delta M = 0$. From (5.13) follows that

$$\kappa_i \;=\; \frac{1}{\delta} \max_{\zeta \in [0,\delta]} \frac{|x_i^*(\zeta) \left(E^*(I+E)^{-*}\tilde{H} + HE\right) x_i(\zeta)|}{|x_i^*(\zeta) H(\zeta) x_i(\zeta)|}.$$

Now, it is obvious that we can not simply derive a bound for $\kappa_i$.

Our first result will be a simple corollary of Theorem 3.1 from section 3.1. But, before we state this result we will repeat some standard notations. Let $G \in \mathbf{C}^{r \times n}$ be matrix with full-row rank, and let

$$G = U \begin{bmatrix} \Sigma & 0 \end{bmatrix} \begin{bmatrix} V_1^* \\ V_0^* \end{bmatrix} \equiv U \Sigma V_1^*,$$

be its HSVD. Similarly, we assume for $\tilde{G}$

$$\tilde{G} \equiv G(I+E) = \tilde{U} \begin{bmatrix} \tilde{\Sigma} & 0 \end{bmatrix} \begin{bmatrix} \tilde{V}_1^* \\ \tilde{V}_0^* \end{bmatrix} \equiv \tilde{U} \tilde{\Sigma} \tilde{V}_1^*.$$

Note that matrices $X = (V^{-1})^*$ and $\tilde{X} = (\tilde{V}^{-1})^*$ perform the diagonalizations (5.2) and (5.4).

Thus, we can use proof of the Theorem 3.1 to obtain bound for eigenvalues of non-singular Hermitian matrix $GJG^*$ perturbed to $G(J + EJ + JE^* + EJE^*)G^*$.

The following theorem contains the relative version of Hoffman-Wielandt type of the perturbation bound.

**Theorem 5.4** *Let $H = G^*G$ and $\tilde{H} = (I+E)^*G^*G(I+E)$ be positive semidefinite where $GJG^*$ and $G(I+E)J(I+E)^*G^*$ are non-singular matrices of order $r$. Let the matrices $X$ and $\tilde{X}$ perform the diagonalizations (5.2) and (5.4), respectively. If $\|E\|_F < 1$, then*

$$\begin{aligned} \sqrt{\sum_{i=1}^{r} \left( \frac{|\tilde{\lambda}_i - \lambda_i|}{\sqrt{|\tilde{\lambda}_i||\lambda_i|}} \right)^2} &\leq \|X_1\| \|\tilde{X}_1\| \cdot \|(I+E) - (I+E)^{-*}\|_F \\ &\leq \|X_1\| \|\tilde{X}_1\| \cdot \frac{2 \cdot \|E\|_F - \|E\|_F \|E\|}{1 - \|E\|}, \end{aligned} \tag{5.44}$$

*provided that right-hand side in (5.44) is less than 2.*

**Proof.** The proof is similar to the proof of Theorem 3.1. Here is $B = G$, $\tilde{B} = \tilde{G}$ and $\widehat{B} = (I + E)$. Therefore $G = U\Sigma V_1^*$, and $\tilde{G} = \tilde{U}\tilde{\Sigma}\tilde{V}_1^*$, where $\Sigma = \operatorname{diag}(|\lambda_1|^{1/2}, \ldots, |\lambda_r|^{1/2})$ and $\tilde{\Sigma} = \operatorname{diag}(|\tilde{\lambda}_1|^{1/2}, \ldots, |\tilde{\lambda}_r|^{1/2})$. From (3.28) and the assumption of theorem it follows that

$$\sum_{i,j=1}^{n} \left( \frac{|\tilde{\lambda}_i - \lambda_i|}{\sqrt{|\tilde{\lambda}_i||\lambda_i|}} \right)^2 \leq \|X_1\|\|\tilde{X}_1\| \cdot \|(I+E)^* - (I+E)^{-1}\|_F,$$

which gives the first inequality from (5.44). The second inequality from (5.44) follows by bounding the right-hand side of the above inequality. ■

Note, that for Hermitian eigenvalue problem

$$Hu_i = \lambda_i u_i,$$

the main difference between additive perturbation bounds and multiplicative perturbation bounds is in extra factor $\|X\|\|\tilde{X}\|$ which appears in additive perturbation bounds of Section 1.2, where $X$ and $\tilde{X}$ are $J$-unitary matrices.

For hyperbolic eigenvalue problem

$$Hx_i = \lambda_i J x_i,$$

the difference between additive perturbation bounds (Theorem 5.2) and multiplicative perturbation bounds is in extra factor $\|X_1\|\|\tilde{X}_1\|$ which appears in multiplicative perturbation bounds. This is a consequence of equivalence between Hermitian eigenvalue problem

$$GJG^* u_i = \lambda_i u_i,$$

and the corresponding hyperbolic eigenvalue problem

$$G^* G x_i = \lambda_i J x_i.$$

Further, we will bound additional term $\|X_1\|\|\tilde{X}_1\|$. The fact that

$$\|X_1\|\|\tilde{X}_1\| \leq \|X\|\|\tilde{X}\|,$$

allow us to derive a bound for $\|\tilde{X}\|$ in the terms of $\|X\|$. Unfortunately, since $G$ and perturbed $\tilde{G} = G(I + E)$ are not quadratic matrices, we can not directly apply Lemma 2.1, to bound $\|\tilde{X}\|$ in terms of $\|X\|$ as we did in Section 5.2.

Thus, we will use the similar approach as it was used in the proof of Theorem 5.2, that is, we will construct matrices $H_0$ and $\tilde{H}_0$ such that both $X^*(H+H_0)X$ and $\tilde{X}^*(\tilde{H}+\tilde{H}_0)\tilde{X}$, are diagonal and $H+H_0$ and $\tilde{H}+\tilde{H}_0$ are both non-singular.

Let the columns of $Q_0$ form the orthonormal basis for $X_0$, that is, $Q_0^*JX_1=0$, and similarly for perturbed quantities.

Let $\begin{bmatrix} X_1 & X_0' \end{bmatrix}$ and $\begin{bmatrix} \tilde{X}_1 & \tilde{X}_0' \end{bmatrix}$ perform diagonalizations

$$\begin{aligned} \begin{bmatrix} X_1^* \\ X_0'^* \end{bmatrix} (H+JQ_0Q_0^*J) \begin{bmatrix} X_1 & X'_0 \end{bmatrix} &= \begin{bmatrix} |\Lambda_r| & \\ & |\Omega_{n-r}| \end{bmatrix} \equiv |\Delta| \\ \begin{bmatrix} X_1^* \\ X_0'^* \end{bmatrix} J \begin{bmatrix} X_1 & X'_0 \end{bmatrix} &= \begin{bmatrix} \widehat{J}_1 & \\ & \widehat{J}_2 \end{bmatrix}, \end{aligned} \tag{5.45}$$

and

$$\begin{aligned} \begin{bmatrix} \tilde{X}_1^* \\ \tilde{X}_0^* \end{bmatrix} (\tilde{H}+J\tilde{Q}_0\tilde{Q}_0^*J) \begin{bmatrix} \tilde{X}_1 & \tilde{X}_0 \end{bmatrix} &= \begin{bmatrix} |\tilde{\Lambda}_r| & \\ & |\tilde{\Omega}_{n-r}| \end{bmatrix} \equiv |\tilde{\Delta}| \\ \begin{bmatrix} \tilde{X}_1^* \\ \tilde{X}_0^* \end{bmatrix} J \begin{bmatrix} \tilde{X}_1 & \tilde{X}'_0 \end{bmatrix} &= \begin{bmatrix} \widehat{J}_1 & \\ & \widehat{J}_2 \end{bmatrix}. \end{aligned} \tag{5.46}$$

Here $X_0' = X_0U_0$, and $U_0$ is the unitary matrix which diagonalizes $X_0^*JQ_0Q_0^*JX_0$, that is,

$$U_0^*X_0^*JQ_0Q_0^*JX_0U_0 = |\Omega_{n-r}|.$$

Note once more that $\|X\| = \| \begin{bmatrix} X_1 & X_0' \end{bmatrix} \|$. Thus, without loss of generality we further assume that $X \equiv \begin{bmatrix} X_+ & X_- \end{bmatrix} = \begin{bmatrix} X_1 & X_0' \end{bmatrix}$. All the above considerations hold analogously for perturbed quantities.

If we set $H_0 = JQ_0Q_0^*J$ and $\tilde{H}_0 = J\tilde{Q}_0\tilde{Q}_0^*J$, then from (5.45) and (5.46) follows that $\begin{bmatrix} X_+ & X_- \end{bmatrix} = \begin{bmatrix} X_1 & X_0' \end{bmatrix}$ and $\begin{bmatrix} \tilde{X}_+ & \tilde{X}_- \end{bmatrix} = \begin{bmatrix} \tilde{X}_1 & \tilde{X}_0' \end{bmatrix}$ simultaneously diagonalize the pairs $(H_{new}, J)$ and $(\tilde{H}_{new}, J)$, respectively, where $H_{new} = H + H_0$ and $\tilde{H}_{new} = \tilde{H} + \tilde{H}_0$.

Our aim is to derive the bound for $\|X_-^*J\tilde{X}_+\|$.

Similarly as in (5.18) for $H_{new}$ and $\tilde{H}_{new}$ we can write

$$\Delta_+X_-^*J\tilde{X}_+ - X_-^*J\tilde{X}_+\tilde{\Delta}_- = -X_-^*\delta H_{new}\tilde{X}_+,$$

where $\delta H_{new} = \delta H + J(\tilde{Q}_0\tilde{Q}_0^* - Q_0Q_0^*)J$. Thus, the above equality can be written as

$$\Delta_+X_-^*J\tilde{X}_+ - X_-^*J\tilde{X}_+\tilde{\Delta}_- = -X_-^*\delta H\tilde{X}_+ - X_-^*\delta H_0\tilde{X}_+, \tag{5.47}$$

where $\delta H_0 = J(\tilde{Q}_0\tilde{Q}_0^* - Q_0Q_0^*)J$.

From the fact that $HX_- = JX_-\Lambda_-$ and $\tilde{H}X_+ = J\tilde{X}_+\tilde{\Lambda}_+$ it follows that

$$X_-\delta H\tilde{X}_+ = X_-^*E^*(I+E)^{-*}J\tilde{X}_+\tilde{\Lambda}_+ + \Lambda_-X_-^*JE\tilde{X}_+, \tag{5.48}$$

where we used the identity $\delta H = E^*(I+E)^{-*}\tilde{H} + HE$. In (5.48) we have $(\Lambda_-)_{ii} \leq 0$ and $(\tilde{\Lambda}_+)_{ii} \geq 0$.

Further, we define a matrix $E_0$, such that $\tilde{G}_0 = G_0(I+E_0)$. This further allows us to write similar equation as (5.48) for $X_-\delta H_0\tilde{X}_+$. For that purpose, note that $GQ_0 = 0$ and $G(I+E)\tilde{Q}_0 = 0$ imply that columns of $Q_0$ form the orthonormal basis for $(I+E)\tilde{Q}_0$.

Let

$$Q_0^*(I+E)\tilde{Q}_0 = U_T\Sigma_TV_T^*,$$

be a SVD of $Q_0^*(I+E)\tilde{Q}_0$. Thus, we can write

$$(I+E)\tilde{S}_0 = S_0\Sigma_T, \tag{5.49}$$

where $S_0 = Q_0U_T$ and $\tilde{S}_0 = \tilde{Q}_0V_T$. Here we have used the fact that

$$Q_0Q_0^*(I+E)\tilde{Q}_0 = (I+E)\tilde{Q}_0,$$

thus $Q_0Q_0^*$ is orthogonal projection onto $\mathcal{R}((I+E)\tilde{Q}_0)$.

If we write $G_0 = S_0^*J$ and $\tilde{G}_0 = \tilde{S}_0^*J$, then

$$H_0 = G_0^*G_0 \quad \text{and} \quad \tilde{H}_0 = \tilde{G}_0^*\tilde{G}_0.$$

Now, we can write

$$\tilde{G}_0 = G_0(I+E_0) \quad \text{where} \quad E_0 = JS_0(\tilde{S}_0^* - S_0^*)J.$$

Note that $\|E_0\| = \|\tilde{S}_0 - S_0\|$. Since $\delta H_0 = E_0^*(I+E_0)^{-*}\tilde{H}_0 + H_0E_0$, we have

$$X_-\delta H_0\tilde{X}_+ = X_-^*E_0^*(I+E_0)^{-*}J\tilde{X}_+\tilde{\Omega}_+ + \Omega_-X_-^*JE\tilde{X}_+, \tag{5.50}$$

where $(\Omega_-)_{ii} \leq 0$ and $(\tilde{\Omega}_+)_{ii} \geq 0$.

Before we derive a bound for $\|X_-^*J\tilde{X}_+\|$ we have to bound $\|E_0\|$ in the terms of $\|E\|$. Note that (5.49) can be interpreted as a singular value decomposition of multiplicative perturbation of unitary matrix $\tilde{S}_0$. By result of Ren-Cang Li [29, (4.6) pp. 965.] it follows that

$$(\Sigma_T)_{ii} - 1 \leq \|E\|.$$

Subtracting $S_0$ from both sides of (5.49) gives

$$\tilde{S}_0 - S_0 = S_0(\Sigma_T - I) - E\tilde{S}_0,$$

which, in turn, gives

$$\|E_0\| = \|\tilde{S}_0 - S_0\| \leq 2\|E\|. \tag{5.51}$$

Now (5.47) can be written as

$$\begin{aligned} \Delta_+ X_-^* J\tilde{X}_+ - X_-^* J\tilde{X}_+\tilde{\Delta}_- \;\; = \;\; & -X_-^* E^*(I+E)^{-*}J\tilde{X}_+\tilde{\Lambda}_+ - \Lambda_- X_-^* JE\tilde{X}_+ \\ & -X_-^* E_0^*(I+E_0)^{-*}J\tilde{X}_+\tilde{\Omega}_+ - \Omega_- X_-^* JE\tilde{X}_+. \end{aligned}$$

Use this component-wise to get

$$\begin{aligned} & ((\Delta_+)_{ii} - (\Delta_-)_{jj})\left(X_-^* J\tilde{X}_+\right)_{ij} = \\ & = -(X_-^*)_{:,i}\left(E^*(I+E)^{-*}J\right)(\tilde{X}_+)_{:,j}(\tilde{\Lambda}_+)_{jj} - (\Lambda_-)_{ii}(X_-^*)_{:,i}JE(\tilde{X}_+)_{:,j} \\ & = -(X_-^*)_{:,j}\left(E_0^*(I+E_0)^{-*}J\right)(\tilde{X}_+)_{:,j}(\tilde{\Omega}_+)_{jj} - (\Omega_-)_{ii}(X_-^*)_{:,i}JE(\tilde{X}_+)_{:,j}, \end{aligned}$$

for all pairs of indices $(i,j)$. For any of $(i,j)$, the eigenvalues $(\tilde{\Lambda}_+)_{jj}$ and $(\Lambda_-)_{ii}$ have opposite signs, or they are equal to zero, and similarly holds for $(\tilde{\Omega}_+)_{jj}$ and $(\Omega_-)_{jj}$. Thus we can write

$$\begin{aligned} |\,[X_-^* J\tilde{X}_+]_{ij}| \;\; \leq \;\; & \frac{1}{2}|X_-^*\left(E^*(I+E)^{-*}J\right)\tilde{X}_+|_{ij} + |X_-^* JE\tilde{X}_+|_{ij} \\ & \frac{1}{2}|X_-^*\left(E_0^*(I+E_0)^{-*}J\right)\tilde{X}_+|_{ij} + |X_-^* JE_0\tilde{X}_+|_{ij} \end{aligned}$$

Here we have used the fact that $\sqrt{ab}/(a+b) \leq 1/2$ for any two positive numbers $a$ and $b$. By squaring the above inequality and adding all terms together for $i = l+1, \cdots, n$ and $j = 1, \cdots, l$, we obtain

$$\|X_-^* J\tilde{X}_+\|_F \leq \frac{1}{2}\|X\|\|\tilde{X}\|\left(\frac{2\|E\|_F - \|E\|\|E\|_F}{1-\|E\|} + \frac{2\|E_0\|_F - \|E_0\|\|E_0\|_F}{1-\|E_0\|}\right).$$

This implies

$$\|X_-^* J\tilde{X}_+\|_F \leq \|X\|\|\tilde{X}\|\varrho, \tag{5.52}$$

where

$$\varrho = \frac{3\|E\|_F - 4\|E\|^2}{1 - 3\|E\| + 2\|E\|^2}. \tag{5.53}$$

This and (2.38) imply that

$$\|\tilde{X}\|_F - \varrho\|X\|^2\|\tilde{X}\| \leq \|X\|\sqrt{1+(\varrho\|X\|\|\tilde{X}\|)^2}. \tag{5.54}$$

If

$$2\cdot\varrho\|X\|^2 < 1,$$

after squaring and rearranging (5.54), we get

$$\|\tilde{X}\|^2 \leq \frac{\|X\|^2}{1-2\varrho\|X\|^2}. \tag{5.55}$$

Now we can present perturbation bound for eigenvalues which contains only unperturbed quantities on the right-hand side. This bound is a simple corollary of Theorem 5.4.

**Corollary 5.4** *Let $H = G^*G$ and $\tilde{H} = (I+E)^*G^*G(I+E)$ be positive semidefinite where $GJG^*$ and $G(I+E)J(I+E)^*G^*$ are non-singular matrices of order $r$. Let the matrix $X$ performs the diagonalizations (5.2). If $2\cdot\varrho\|X\|^2 < 1$, where $\varrho$ is defined by (5.53), then*

$$\sqrt{\sum_{i=1}^{r}\left(\frac{|\tilde{\lambda}_i-\lambda_i|}{\sqrt{|\tilde{\lambda}_i||\lambda_i|}}\right)^2} \leq \frac{\|X\|^2}{\sqrt{1-2\varrho\|X\|^2}}\cdot\frac{2\cdot\|E\|_F-\|E\|_F\|E\|}{1-\|E\|}, \tag{5.56}$$

*provided that right-hand side in (5.56) is less than 2.*

**Proof.** The bound (5.56) is obtained by inserting (5.55) in (5.44). ■

We conclude this section by proving an eigenvector bound for the multiplicative perturbations. But before we state our result we will need some additional notations. Let $X$ perform the diagonalization (5.37). We will decompose $X$ as $X = \begin{bmatrix} X_{11} & X_{20}\end{bmatrix}$, where $X_{20} = \begin{bmatrix} X_{12} & X_0\end{bmatrix}$, so that

$$\begin{aligned}
\begin{bmatrix} {X_{11}}^* \\ X_{20}^*\end{bmatrix} H \begin{bmatrix} X_{11} & X_{20}\end{bmatrix} &= \begin{bmatrix} |\Lambda_{11}| & \\ & |\Lambda_2|\end{bmatrix}, \\
\begin{bmatrix} {X_{11}}^* \\ X_{20}^*\end{bmatrix} J \begin{bmatrix} X_{11} & X_{20}\end{bmatrix} &= \begin{bmatrix} \widehat{J}_{11} & \\ & \widehat{J}_{20}\end{bmatrix},
\end{aligned}$$

where

$$\Lambda_2 = \begin{bmatrix} \Lambda_{22} & \\ & 0\end{bmatrix}, \qquad \widehat{J}_{20} = \begin{bmatrix} \widehat{J}_{22} & \\ & \widehat{J}_2\end{bmatrix}.$$

We assume similar decomposition for $\tilde{X} = \begin{bmatrix} \tilde{X}_{11} & \tilde{X}_{20}\end{bmatrix}$.

**Theorem 5.5** *Let $H + \delta H = (I + E)^* H (I + E)$, and let the matrices $X$ and $\tilde{X}$ perform the diagonalizations (5.2) and (5.4), respectively. Let the relative gap between the sets of eigenvalues from $\tilde{\Lambda}_{11}$ and $\Lambda_2$ be defined as*

$$\overline{\mathrm{rg}}(\tilde{\Lambda}_{11}, \Lambda_2) = \min_{i,j} \frac{|\,[\tilde{\Lambda}_{11}]_{ii} - [\Lambda_2]_{jj}|}{\max\{|\,[\tilde{\Lambda}_{11}]_{ii}|, |\,[\Lambda_2]_{jj}|\}}. \tag{5.57}$$

*Let $\varrho$ be defined by (5.53). If $\varrho < 1/(2\|X\|^2)$, then*

$$\|\sin\Theta(X_{11}, \tilde{X}_{11})\|_F \leq \frac{\|X\|^4}{1 - 2\varrho\,\|X\|^2} \cdot \frac{2\|E\|_F - \|E\|\|E\|_F}{1 - \|E\|} \cdot \frac{1}{\overline{\mathrm{rg}}(\tilde{\Lambda}_1, \Lambda_2)}. \tag{5.58}$$

*In particular, if $H = DAD$, where $H$ and $A$ are positive definite, and $D = \mathrm{diag}\,(\sqrt{H_{ii}})$, then*

$$\|\sin\Theta(X_{11}, \tilde{X}_{11})\|_F \leq \frac{\kappa(A)}{1 - 2\alpha\sqrt{\kappa(A)}} \cdot \frac{2\|E\|_F - \|E\|\|E\|_F}{1 - \|E\|} \cdot \frac{1}{\overline{\mathrm{rg}}(\tilde{\Lambda}_1, \Lambda_2)}. \tag{5.59}$$

**Proof.** The proof is a combination of the proofs of Theorems 5.3 and Theorem 5.4. As in the proof of Theorem 5.3, the relations (5.36), (5.41) and (5.42) imply

$$\|\sin\Theta(X_{11}, \tilde{X}_{11})\|_F \leq \|X\|\,\|\tilde{X}\|\,\|X_{20}^* J \tilde{X}_{11}\|_F. \tag{5.60}$$

Using $\delta H = E^*(I+E)^{-*}\tilde{H} + HE$ and interpreting the relation (5.18) block-wise we get

$$\Lambda_2 X_{20}^* J \tilde{X}_{11} - X_{20}^* J \tilde{X}_{11} \tilde{\Lambda}_{11} = X_{20}^* E^* (I+E)^{-*} J \tilde{X}_{11} \tilde{\Lambda}_{11} + \Lambda_2 X_{20}^* J E \tilde{X}_{11}.$$

Writing the upper equality component-wise, we get

$$|X_{20}^* J \tilde{X}_1|_{ij} \leq (|X_2^* \widehat{E} J \tilde{X}_1|_{ij} + |X_2^* J E \tilde{X}_1|_{ij}) \cdot \frac{1}{\overline{\mathrm{rg}}(\tilde{\Lambda}_1, \Lambda_2)}.$$

Therefore,

$$\begin{aligned} \|X_2^* J \tilde{X}_1\|_F &\leq \frac{\|X_2^* \widehat{E} J \tilde{X}_1\|_F + \|X_2^* J E \tilde{X}_1\|_F}{\overline{\mathrm{rg}}(\tilde{\Lambda}_1, \Lambda_2)} \\ &\leq \|X\|\,\|\tilde{X}\| \cdot \frac{2\|E\|_F - \|E\|\|E\|_F}{1 - \|E\|} \cdot \frac{1}{\overline{\mathrm{rg}}(\tilde{\Lambda}_1, \Lambda_2)}. \end{aligned}$$

Now (5.58) follows by inserting this and (5.55) into (5.60). Finally, (5.59) follows from (5.58) and (2.35). ∎

The following numerical example illustrates the quality of bounds (5.56) and (5.56).

**Example 5.2** *Let $H = G^*G$, where*

$$G = \begin{bmatrix} 6.945\cdot 10^2 & 1.014\cdot 10^3 & 9.937\cdot 10^2 & 3.020\cdot 10^3 & 1.361\cdot 10^3 & 9.941\cdot 10^2 \\ 1.375 & 6.721\cdot 10 & 4.006\cdot 10 & 8.386\cdot 10 & 4.194\cdot 10 & 3.768\cdot 10 \\ 6.770\cdot 10^{-1} & 4.201\cdot 10^{-1} & 1.621\cdot 10^{-1} & 5.377\cdot 10^{-1} & 6.705\cdot 10^{-1} & 1.571\cdot 10^{-2} \\ 6.813\cdot 10^{-2} & 3.795\cdot 10^{-2} & 8.318\cdot 10^{-2} & 5.028\cdot 10^{-2} & 7.095\cdot 10^{-2} & 4.289\cdot 10^{-2} \end{bmatrix},$$

*and let $J = \operatorname{diag}(1,1,-1,-1,1,-1)$. The non-zero eigenvalues of the pair $(H,J)$ are (properly rounded)*

$$\lambda_1 = 1.619\cdot 10^3,\quad \lambda_2 = 7.607\cdot 10^{-1},\quad \lambda_3 = -6.396\cdot 10^{-3},\quad \lambda_4 = -7.733\cdot 10^6.$$

*Let the perturbed matrix $\tilde{H}$ be given as in Theorem 5.4 with*

$$E = 10^{-6}\cdot \begin{bmatrix} 0.3 & 0.5 & 0.5 & 0.5 & 0.4 & 0.1 \\ 0.5 & 0.3 & 0.6 & 0.6 & 0.3 & 0.8 \\ 0.1 & 0.4 & 0.2 & 0.8 & 0.9 & 0.4 \\ 0.1 & 0.2 & 0.4 & 0.1 & 0.4 & 0.5 \\ 0.6 & 0.6 & 0.8 & 0.6 & 0.8 & 0.2 \\ 0.4 & 0.8 & 0.7 & 0.1 & 0.1 & 0.6 \end{bmatrix}.$$

*When applying Theorem 5.4, the left hand side of (5.44) (the exact perturbation) is equal to $3.663\cdot 10^{-6}$ and the right hand side in the first inequality from bound (5.44) is $2.194\cdot 10^{-5}$. The right hand side from (5.56) is $2.482\cdot 10^{-5}$. (here $\varrho = 1.052\cdot 10^{-5}$ and $\|X\|^2 = 3.539$.) On the other hand, the bounds of [46, Corollary VI.3.3] are here useless. Indeed, since there $\|E\| = 0$ but $\|F\| = \|\delta H\| = 8.083$ and $\gamma(A,B) = 0.866$ according to [46, Definition VI.1.17], the assumption of [46, Corollary VI.3.3] is not fulfilled.*

*Let us now consider invariant subspaces. The bound of Theorem 5.5 is obtained by simply dividing the eigenvalue bound (5.56) by the relative gap (5.57). Let $\sin\Theta_i = \sin\Theta(x_i,\tilde{x}_i)$, where $x_i$ is the eigenvector of $\lambda_i$. Table 5.2 gives the corresponding relative gaps for non zero eigenvalues, the exact perturbations and the bound (5.58).*

| *i* | *1* | *2* | *3* | *4* |
|---|---|---|---|---|
| $\overline{\text{rg}}(\tilde{\lambda}_i,\Lambda_2)$ | 1 | 0.995 | 0.995 | 1 |
| $\|\sin\Theta_i\|_F$ | $1.449\cdot 10^{-8}$ | $1.810\cdot 10^{-8}$ | $1.095\cdot 10^{-7}$ | $1.089\cdot 10^{-7}$ |
| *Theorem 5.5* | $2.482\cdot 10^{-5}$ | $2.494\cdot 10^{-5}$ | $2.494\cdot 10^{-5}$ | $2.482\cdot 10^{-5}$ |

Table 5.2: Eigenvector perturbations

# Chapter 6

# Generalized eigenvalue problem

In this section we consider the generalized eigenvalue problem

$$Hx = \lambda Mx, \tag{6.1}$$

where $H, M \in \mathbf{C}^{n\times n}$ are Hermitian and $x \in \mathbf{C}^n$, $x \neq 0$.

Our bounds will be given for diagonalizable matrix pairs $(H, M)$ for which there exists a non-singular matrix $X$ such that

$$X^*HX = \Omega, \quad X^*MX = J,$$

where

$$\begin{aligned} \Omega = \operatorname{diag}(\omega_1, \ldots, \omega_n) \qquad & J = \operatorname{diag}(j_1, \ldots, j_n), \\ \omega_i \in \mathbf{R} \qquad & j_i \in \{-1, 1\}, \end{aligned}$$

for $i = 1, \ldots, n$. Before we present our results, we give a short overview on existing results.

This chapter is organized as follows. First we state some existing classical and relative results. Classical eigenvalue bounds due to Stewart and Sun [46, Chapter VI] are given in Theorem 6.1 and Corollary 6.1, while the classical version of the $\sin\Theta$ theorem for matrix pair is given in Theorem 6.2 [46, Theorem VI.3.9]. Further, in Theorem 6.3 and Corollary 6.2 we give local optimal relative perturbation bound for eigenvalues due to Barlow and Slapničar [2]. In Section 6.1 we derive new relative Hoffman–Wielandt type bound for eigenvalues, and state the Weyl-type relative perturbation bound by Dhillon [9]. Finally, in Section 6.2 we derive a relative version of the $\sin\Theta$ theorem, and give a numerical example.

Stewart and Sun [46, Chapter VI] developed out a perturbation theory for Hermitian pairs $(H, M)$ that satisfy

$$\gamma = \min_{\substack{x \in \mathbf{C}^n \\ \|x\|=1}} |x^*(H + \imath M)x| = \min_{\substack{x \in \mathbf{C}^n \\ \|x\|=1}} \sqrt{(x^*Hx)^2 + (x^*Mx)^2} > 0. \tag{6.2}$$

The pair $(H, M)$ which satisfies (6.2) is called definite pair. Since pair $(H, M)$ can have infinite eigenvalue (that is, $\lambda = \infty$) Stewart and Sun [46] use the following notation for the eigenvalue problem (6.1)

$$\beta Hx = \alpha Mx. \tag{6.3}$$

The pair $(\alpha, \beta)$ corresponds to the eigenvalue $\lambda$ if $\lambda = \alpha/\beta$. Moreover the infinite eigenvalue corresponds to nonzero pairs $(\alpha, \beta)$ for which $\beta = 0$, which is a case that is not essentially different from the case when $\alpha = 0$, $\beta \neq 0$, that is, $\lambda = 0$.

The distance between the pairs $(\alpha, \beta)$ and $(\gamma, \delta)$ is measured using the chordal distance. The chordal distance between the pairs $(\alpha, \beta)$ and $(\gamma, \delta)$ is defined ([46, Definition 1.20. VI] or [37, pg. 342]) as

$$\chi((\alpha, \beta), (\gamma, \delta)) = \frac{|\alpha\delta - \beta\gamma|}{\sqrt{\alpha^2 + \beta^2}\sqrt{\gamma^2 + \delta^2}}.$$

If we set $\lambda = \alpha/\beta$ and $\mu = \gamma/\delta$, then we have

$$\chi(\lambda, \mu) = \frac{|\lambda - \mu|}{\sqrt{1 + \lambda^2}\sqrt{1 + \mu^2}}.$$

Let $(H, M)$ be definite pair, and let $(\tilde{H}, \tilde{M}) = (H + \delta H, M + \delta M)$ be its perturbation. The following theorem is a simple corollary of the [46, Theorem 3.2. VI] and it bounds the chordal distance between the pairs $(\alpha_i, \beta_i)$ and $(\tilde{\alpha}_i, \tilde{\beta}_i)$ which corresponds to $i$-th eigenvalues of the pairs $(H, M)$, and $(\tilde{H}, \tilde{M})$, respectively. The bound is given in terms of $\|\delta H\|$ and $\|\delta M\|$.

**Theorem 6.1 (Stewart and Sun)** *If*

$$\|\delta H\|^2 + \|\delta M\|^2 < \gamma^2(H, M)$$

*then the pair* $(\tilde{H}, \tilde{M}) \equiv (H + \delta H, M + \delta M)$ *is definite and*

$$\chi((\alpha, \beta), (\tilde{\alpha}_i, \tilde{\beta}_i)) \;\leq\; \frac{\sqrt{\|\delta H\|^2 + \|\delta M\|^2}}{\gamma(H, M)}, \quad i = 1, \ldots, n. \tag{6.4}$$

**Proof.** See Theorem VI.3.2. and Corollary VI.3.3. from [46]. ∎

Especially, if $M$ is positive definite a pair $(H, M)$ is called symmetric definite (see [19] or [37]).

Let $(H, M)$ be a symmetric definite pair with eigenvalues $\lambda_i = \alpha_i/\beta_i$, and let $(\tilde{H}, \tilde{M}) = (H + \delta H, M + \delta M)$ be the corresponding perturbed pair with eigenvalues $\tilde{\lambda}_i = \tilde{\alpha}_i/\tilde{\beta}_i$, $i = 1, \ldots, n$. Using the simple equality

$$\chi(\tilde{\lambda}_i, \lambda_i) = \frac{|\tilde{\lambda}_i - \lambda_i|}{\sqrt{1+\tilde{\lambda}_i^2}\sqrt{1+\lambda_i^2}} \quad = \quad \frac{|\tilde{\lambda}_i - \lambda_i|}{\sqrt{(1+\tilde{\lambda}_i\lambda_i)^2 + (\tilde{\lambda}_i - \lambda_i)^2}}, \tag{6.5}$$

we have the following corollary (the similar, but something sharper bound is presented in [46, Corollary VI.3.3] or [19, Theorem 8.7.3]) of Theorem 6.1.

**Corollary 6.1** *Let $(H, M)$ be a symmetric definite pair with eigenvalues $\lambda_1 \geq \ldots \geq \lambda_n$. Let $(\tilde{H}, \tilde{M}) \equiv (H + \delta H, M + \delta M)$ be corresponding perturbed pair with eigenvalues $\tilde{\lambda}_1 \geq \ldots \geq \tilde{\lambda}_n$. If*

$$\|\delta H\|^2 + \|\delta M\|^2 < \gamma^2(H, M),$$

*then the pair $(\tilde{H}, \tilde{M})$ is definite and*

$$|\arctan \tilde{\lambda}_i - \arctan \lambda_i| \quad \leq \quad \arctan \frac{\varepsilon}{\sqrt{\gamma^2(H, M) - \varepsilon^2}}, \qquad i = 1, \ldots, n, \tag{6.6}$$

*where $\varepsilon = \sqrt{\|\delta H\|^2 + \|\delta M\|^2}$.*

**Proof**. Using (6.5) the bound (6.4) can be written as

$$\frac{(\tilde{\lambda}_i - \lambda_i)^2}{(1+\tilde{\lambda}_i\lambda_i)^2 + (\tilde{\lambda}_i - \lambda_i)^2} \leq \frac{\varepsilon^2}{\gamma^2(H, M)}.$$

Using this and the equality that

$$|\arctan \tilde{\lambda}_i - \arctan \lambda_i| = \arctan \frac{|\tilde{\lambda}_i - \lambda_i|}{1 + \tilde{\lambda}_i\lambda_i},$$

the bound (6.6) follows after a simple manipulation. ∎

Stewart and Sun [46, Chapter VI] also have worked out perturbation bounds for the eigenspace of a regular matrix pair $(H, M)$. By [46, Definition 1.2] a pair $(H, M)$ is regular if there is $\alpha$ and $\beta$ such a that the matrix $\beta H - \alpha M$ is non-singular.

For the sake of simplicity, in this introduction we describe only the perturbation bound for the sine of the canonical angles between eigenspaces of symmetric definite pair $(H, M)$ perturbed to $(H + \delta H, M + \delta M)$.

Let $(H, M)$ be a symmetric definite pair. There exists non-singular matrix ([19, Corollary 8.7.2] or [46, Theorem 3.5.VI]) $X = \begin{bmatrix} X_1 & X_2 \end{bmatrix}$, which simultaneously diagonalizes the pair $(H, M)$, that is,

$$\begin{bmatrix} X_1^* \\ X_2^* \end{bmatrix} H \begin{bmatrix} X_1 & X_2 \end{bmatrix} = \begin{bmatrix} \Lambda_1 & \\ & \Lambda_2 \end{bmatrix} \quad \begin{bmatrix} X_1^* \\ X_2^* \end{bmatrix} M \begin{bmatrix} X_1 & X_2 \end{bmatrix} = \begin{bmatrix} \Omega_1 & \\ & \Omega_2 \end{bmatrix}, \tag{6.7}$$

where

$$\begin{aligned} \Lambda_1 &= \operatorname{diag}(\lambda_1, \ldots, \lambda_k), & \Lambda_2 &= \operatorname{diag}(\lambda_{k+1}, \ldots, \lambda_n), \\ \Omega_1 &= \operatorname{diag}(\omega_1, \ldots, \omega_k), & \Omega_2 &= \operatorname{diag}(\omega_{k+1}, \ldots, \omega_n), \end{aligned}$$

and $X_1 \in \mathbf{C}^{n \times k}$, $X_2 \in \mathbf{C}^{n \times n-k}$. Let $\varrho(\Lambda_1, \Omega_1)$ and $\varrho(\Lambda_2, \Omega_2)$ denote the set of eigenvalues of the pair $(\Lambda_1, \Omega_1)$ and $(\Lambda_2, \Omega_2)$, respectively. If $\varrho(\Lambda_1, \Omega_1) \cap \varrho(\Lambda_2, \Omega_2) = \emptyset$, then the columns of $X_1$ span simple eigenspace of the pair $(H, M)$. The following theorem contains a bound for the Frobenius norm of the diagonal matrix which contains the sines of the canonical angles between eigenspace $\mathcal{R}(X_1)$ and corresponding perturbed eigenspace $\mathcal{R}(\tilde{X}_1)$.

**Theorem 6.2 (Sun)** *Let the definite pair $(H, M)$ be decomposed as in (6.7) where $X_1$ and $X_2$ have orthonormal columns. Let the analogous decomposition be given for the pair $(\tilde{H}, \tilde{M}) \equiv (H + \delta H, M + \delta M)$. If*

$$\delta = \min\{\chi(\tilde{\lambda}, \lambda) : \lambda \in \varrho(\Lambda_1, \Omega_1), \tilde{\lambda} \in \varrho(\tilde{\Lambda}_2, \tilde{\Omega}_2)\},$$

*then*

$$\|\sin \Theta[\mathcal{R}(X_1), \mathcal{R}(\tilde{X}_1)]\|_F \leq \frac{\sqrt{\|H^2 + M^2\|}}{\gamma(H, M)\gamma(\tilde{H}, \tilde{M})} \frac{\sqrt{\|\delta H X_1\|_F^2 + \|\delta M X_1\|_F^2}}{\delta}. \tag{6.8}$$

**Proof**. See [46, Theorem VI.3.9.] and [47, 48]. ■

All the above results belong to the so called classical or absolute perturbation theory for matrix pairs $(H, M)$. The main characteristics of this theory were already discussed in the case of single matrix in Section 1.1.

On the other hand, there are several results in the last ten years which consider the perturbations of eigenvalues and corresponding eigenvectors using the relative approach.

We have already mentioned some of these results. Theorem 1.2 was presented in [55] and it bounds perturbations of eigenvalues of the symmetric definite pair $(H, M)$. Theorem 5.1 from [2] bounds perturbations of eigenvalues of the Hermitian pair $(H, M)$, where only the existence of non-singular matrix $X$ which simultaneously diagonalizes the pair $(H, M)$, is assumed.

Moreover, Barlow and Slapničar in [2] have derived the condition number $\kappa_i$ from Theorem 5.1 in the case when the pair $(H, M)$ is perturbed to $(H + \delta H, M + \delta M)$, and $M$ is positive semidefinite.

In addition to Theorem 5.1, Theorem 6.3 and Corollary 6.2, we presented below, give local relative perturbation bounds for eigenvalues under structured perturbations. Such perturbations also include floating-point perturbations described in Section 1.

Now we shortly describe these results.
We assume that the family of pencils $(H(\zeta) - \lambda M(\zeta))$, $\zeta \in [0, \delta]$, defined by (5.9) has the form

$$H(\zeta) = X^{-*}(\zeta)\Lambda(\zeta)X^{-1}(\zeta), \qquad M(\zeta) = X^{-*}(\zeta)J(\zeta)X^{-1}(\zeta),$$

where

$$\begin{array}{rcl} X(\zeta) & = & [\, x_1(\zeta), \ldots, x_n(\zeta)\,] \\ \Lambda(\zeta) & = & \operatorname{diag}(\lambda_1(\zeta), \ldots, \lambda_n(\zeta)) \\ J(\zeta) & = & \operatorname{diag}(j_1(\zeta), \ldots, j_n(\zeta)), \end{array}$$

where

$$j_i(\zeta) = \begin{cases} 0 & if \quad x_i(\zeta) \in \operatorname{Null}(M) \\ 1 & \qquad otherwise \end{cases} .$$

Further, we define the spectral absolute value of the matrix $H(\zeta)$ with respect to $M$ as the matrix $H_M^{\ddagger}(\zeta)$ given by

$$H_M^{\ddagger}(\zeta) = X^{-*}(\zeta)|\Lambda(\zeta)|X^{-1}(\zeta).$$

Here $|\Lambda(\zeta)| = \operatorname{diag}(|\lambda_1(\zeta)|, |\lambda_2(\zeta)|, \ldots, |\lambda_n(\zeta)|)$. If $M = I$, then we define $H^{\ddagger}(\zeta)$ by

$$H^{\ddagger}(\zeta) = H_I^{\ddagger}(\zeta).$$

If we let $X^{-1}(\zeta)$ have the orthogonal factorization,

$$X^{-1}(\zeta) = Q(\zeta)R(\zeta)$$

where $Q(\zeta)$ is unitary and $R(\zeta)$ is upper triangular, then it is easily seen that

$$H_M^{\ddagger}(\zeta) = R^*(\zeta)(R^{-*}(\zeta)H(\zeta)R^{-1}(\zeta))^{\ddagger}R(\zeta). \tag{6.9}$$

Equation (6.9) is the definition given by Veselić and Slapničar [55] for the case where $M$ is nonsingular. We also note that for the case $M = I$, we have

$$H^{\ddagger}(\zeta) = \sqrt{H^2(\zeta)}.$$

Now, we will define a truncated version of $H_M^{\ddagger}(\zeta)$.

Let $\mathcal{S}$ be defined by (5.11). Then, we define $H_M^{\ddagger}(\zeta;\mathcal{S}), \zeta \in [0,\delta]$ as the truncated spectral absolute value of $H(\zeta)$ with respect to $M(\zeta)$. It is the matrix $H_M^{\ddagger}(\zeta;\mathcal{S})$ such that

$$H_M^{\ddagger}(\zeta;\mathcal{S}) = X^{-*}|\Lambda(\zeta;\mathcal{S})|X^{-1}$$

where

$$|\Lambda(\zeta;\mathcal{S})| = \operatorname{diag}(|\lambda_1(\zeta;\mathcal{S})|, \ldots, |\lambda_n(\zeta;\mathcal{S})|)$$

and

$$\lambda_i(\zeta;\mathcal{S}) = \begin{cases} 0 & \text{if } i \in \mathcal{S} \\ \lambda_i(\zeta) & \text{otherwise} \end{cases}$$

We define $M(\zeta;\mathcal{S})$ and $J(\zeta;\mathcal{S})$ conformally.

Matrix $H_M^{\ddagger}(\zeta;\mathcal{S})$ is positive semi-definite. We can factor both $H_M(\zeta;\mathcal{S})$ and $M(\zeta;\mathcal{S})$ into the form

$$H_M^{\ddagger}(\zeta;\mathcal{S}) = C(\zeta;\mathcal{S})^* C(\zeta;\mathcal{S}) \tag{6.10}$$

and

$$M(\zeta;\mathcal{S}) = G(\zeta;\mathcal{S})^* G(\zeta;\mathcal{S}) \tag{6.11}$$

where

$$C(\zeta;\mathcal{S}) = U(\zeta)\Phi(\zeta;\mathcal{S})X^{-1}(\zeta) \in \mathbf{C}^{m\times n}, m \geq n \tag{6.12}$$

$$G(\zeta;\mathcal{S}) = V(\zeta)J(\zeta)X^{-1}(\zeta) \in \mathbf{C}^{p\times n}, p \geq n. \tag{6.13}$$

In (6.12) and (6.13), $U(\zeta)$ and $V(\zeta)$ are matrices with orthonormal columns. Here $\Phi(\zeta;\mathcal{S})$ satisfies

$$\Phi(\zeta;\mathcal{S}) = \operatorname{diag}(\phi_i(\zeta;\mathcal{S})) = \sqrt{|\Lambda(\zeta;\mathcal{S})|}.$$

The form (6.12) and (6.13) describes the quotient singular value decomposition (QSVD) of the pair $(C(\zeta;\mathcal{S}), G(\zeta;\mathcal{S}))$ [31, 36]. This allows us to establish bounds that are similar to those in the previous section.

Clearly, the $G(\zeta;\mathcal{S})$-weighted psuedoinverse of $C(\zeta;\delta)$ is given by

$$C_G^{\dagger}(\zeta;\mathcal{S}) \equiv X(\zeta)\Phi^{\dagger}(\zeta;\mathcal{S})U^*(\zeta).$$

Likewise, the $C(\zeta;\mathcal{S})$-weighted psuedoinverse of $G(\zeta;\delta)$ is

$$G_C^{\dagger}(\zeta;\mathcal{S}) \equiv X(\zeta)J^{\dagger}(\zeta;\mathcal{S})V^*(\zeta).$$

Using this structure, we can establish bounds on all of the eigenvalues that do not change sign under the perturbation.

**Theorem 6.3 (Barlow and Slapničar)** *[2, Theorem 3.6] Let the pair $(\tilde{H},\tilde{M})$ be defined by (5.8). For $\zeta\in[0,\delta]$ let $(\lambda_i(\zeta),x_i(\zeta))$ be the ith eigenpair of the pair $(H(\zeta),M(\zeta))$ defined by (5.9). Let $C(\zeta;\mathcal{S})$ and $G(\zeta;\mathcal{S})$ be defined as in (6.10) and (6.11), respectively, and let the QSVD of $(C(\zeta;\mathcal{S}),G)$ be given by (6.12)-(6.13). Define $\mathcal{S}$ as in (5.11). Then for each $\lambda_i(\zeta), i\in\mathcal{S}$, satisfies*

$$\exp(-\delta\kappa_i) \leq \frac{\lambda_i(\delta)}{\lambda_i(0)} \leq \exp(\delta\kappa_i),$$

*where*

$$\begin{aligned}\kappa_i &\leq \max_{\zeta\in[0,\delta]}\frac{|x_i^*(\zeta)Ex_i^*(\zeta))|}{x_i^*(\zeta)H_M^{\dagger}(\zeta;\mathcal{S})x_i^*(\zeta)}+\max_{\zeta\in[0,\delta]}\frac{|x_i^*(\zeta)Fx_i^*(\zeta))|}{x_i^*(\zeta)M(\zeta;\mathcal{S})x_i^*(\zeta)} \qquad (6.14)\\ &= \max_{\zeta\in[0,\delta]}|u_i^*(\zeta)[C_G^{\dagger}(\zeta;\mathcal{S})]^*EC_G^{\dagger}(\zeta;\mathcal{S})u_i(\zeta)|+\max_{\zeta\in[0,\delta]}|v_i^*(\zeta)[G_C^{\dagger}(\zeta;\mathcal{S})]^*FG_C^{\dagger}(\zeta;\mathcal{S})v_i(\zeta)|\\ &\leq \max_{\zeta\in[0,\delta]}\|[C_G^{\dagger}(\zeta;\mathcal{S})]^*EC_G^{\dagger}(\zeta;\mathcal{S})\|+\max_{\zeta\in[0,\delta]}\|[G_C^{\dagger}(\zeta;\mathcal{S})]^*FG_C^{\dagger}(\zeta;\mathcal{S})\|.\end{aligned}$$

**Proof:** See Theorem 3.6. from [2].

The following corollary gives the bound on special types of structured perturbations. The first a scaled perturbation of the type discussed by Barlow and Demmel [1]. Here $E$ has the form

$$E=D_H^*E_SD_H,\quad \|E_S\|\leq 1,\quad F=D_M^*F_SD_M,\quad \|F_S\|\leq 1. \qquad (6.15)$$

**Corollary 6.2 (Barlow and Slapničar)** *[2, Corollary 3.7] Assume the hypothesis and terminology of Theorem 6.3. Assume that $E$ and $F$ has the form (6.15). Then for each $i\in\mathcal{S}$*

$$\begin{aligned}\kappa_i &\leq \max_{\zeta\in[0,\delta]}\|D_HC_G^{\dagger}(\zeta;\mathcal{S})u_i(\zeta)\|^2+\max_{\zeta\in[0,\delta]}\|D_MG_C^{\dagger}(\zeta;\mathcal{S})v_i(\zeta)\|^2.\\ &\leq \max_{\zeta\in[0,\delta]}\|D_HC_G^{\dagger}(\zeta;\mathcal{S})\|^2+\max_{\zeta\in[0,\delta]}\|D_MG_C^{\dagger}(\zeta;\mathcal{S})\|^2.\end{aligned}$$

Similarly as in Corollary 5.2 the above bound is local and the first bound for $\kappa_i$ is nearly optimal. However, maximum cannot be readily computed. We can compute only the first order approximation for $\delta=0$. Thus, we need another bound which is applicable and simple to compute. Such a bound is derived in the following section.

## 6.1 Eigenvalue perturbation bounds

In this section we present the perturbation bounds for eigenvalues of the Hermitian pair $(H,M)$ where $H$ is positive semi-definite and $M$ is non-singular. This result is application of the results from the results described in Section 5 for the hyperbolic eigenvalue problem of the Hermitian pair $(H,M)$. But, before we continue, we will briefly described the main idea of our approach to this problem.

Let $(H,M)$ be Hermitian pair, where $H,M \in \mathbf{C}^{n\times n}$, and $H$ is positive semi-definite with $\operatorname{rank}(H)=r$, and $M$ is non-singular. We will decompose $H$ and $M$ as

$$\begin{aligned} H &= G^*G \qquad G\in\mathbf{C}^{r\times n},\ \operatorname{rank}(\mathrm{G})=r\\ M &= L^*JL, \qquad J=\operatorname{diag}(\pm 1). \end{aligned} \tag{6.16}$$

Additionally, we will assume that $GM^{-1}G^*$ is non-singular, which will allow as to apply HSVD on the matrix $S=GL^{-1}$. The HSVD of $S$ can be written as

$$S \equiv GL^{-1} \;=\; U\,[\,\Sigma \quad 0\,]\,F^{-1}. \tag{6.17}$$

where $F$ is $J$-unitary and the eigenvalue problem for the pair $(H,M)$ can be considered as the hyperbolic eigenvalue problem for the pair $(S^*S,J)$. The bounds obtained by this approach have the same structure as the bound for hyperbolic case with additional factor $\|\delta LL^{-1}\|$, which is a measure for perturbation of decomposition $M=L^*JL$.

We consider a perturbations of the Hermitian pair $(H,M)$ where

$$H=D^*AD,\quad D\in\mathbf{C}^{r\times n},\quad \operatorname{rank}(D)=r,\quad A\in\mathbf{C}^{r\times r}, \tag{6.18}$$

where $A$ is positive definite and $M$ is non-singular.
Let the pair $(H,M)$ be perturbed to $(\tilde{H},\tilde{M})=(H+\delta H, M+\delta M)$ where

$$\tilde{H}=D^*(A+\delta A)D,\qquad \tilde{M}=(L+\delta L)^*J(L+\delta L). \tag{6.19}$$

We also assume that in (6.19) $A+\delta A$ is positive definite, and $L+\delta L$ is non-singular, with corresponding dimensions.

If we write

$$G \;=\; A^{1/2}D. \tag{6.20}$$

then $H$ can be written as

$$H \;=\; G^*G. \tag{6.21}$$

Now, if $GM^{-1}G^*$ is non-singular then from the HSVD of the matrix $S = GL^{-1}$ follows that

$$S = U\,[\,\Sigma \quad 0\,]\,F^{-1}.$$

Thus, the matrix $X = L^{-1}F = \begin{bmatrix} X_1 & X_0 \end{bmatrix}$ simultaneously diagonalizes the pair $(H, M)$. Indeed, it is easy to show that

$$X^*HX = \begin{bmatrix} \Sigma^2 & 0 \\ 0 & 0 \end{bmatrix} = |\Lambda| \qquad X^*MX = \begin{bmatrix} J_1 & 0 \\ 0 & J_2 \end{bmatrix} = \widehat{J}, \tag{6.22}$$

where $\widehat{J}$ is diagonal matrix with $\pm 1$ on its diagonal, and possibly different from $J$, that is, $\widehat{J}$ is a permutation of $J$. Similar holds for perturbed quantities, assuming that $\tilde{G}\tilde{M}^{-1}\tilde{G}^*$ is non-singular, where

$$\tilde{G} = \tilde{A}^{1/2}D, \quad \text{and} \quad \tilde{A} \equiv A + \delta A. \tag{6.23}$$

This means that there exists a non-singular matrix $\tilde{X} = (L + \delta L)^{-1}\tilde{F} = \begin{bmatrix} \tilde{X}_1 & \tilde{X}_0 \end{bmatrix}$ such that

$$\tilde{X}^*\tilde{H}\tilde{X} = \begin{bmatrix} \tilde{\Sigma}^2 & 0 \\ 0 & 0 \end{bmatrix} = |\tilde{\Lambda}| \qquad \tilde{X}^*\tilde{M}\tilde{X} = \begin{bmatrix} J_1 & 0 \\ 0 & J_2 \end{bmatrix} = \widehat{J}, \tag{6.24}$$

Note that the $i$-th column of $X$ is the $i$-th eigenvector which corresponds to the $i$-th eigenvalue $\lambda_i = |\Lambda|_{ii}\widehat{J}_{ii}$ of the pair $(H, M)$. Similar holds for perturbed quantities.

We present two perturbation bounds for eigenvalues. The first bound is relative version of the Hoffman–Wielandt type of bound, and it is obtained by using similar approach as in Theorem 5.2 and 5.4. The second bound is relative version of the Weyl type of bound. This bound is obtained by using result (1.32) from Chapter 1.

**Theorem 6.4** *Let $(H, M)$ be Hermitian pair as in (6.18), and let $(\tilde{H}, \tilde{M})$ be as in (6.19). Let $GM^{-1}G^*$ and $\tilde{G}\tilde{M}^{-1}\tilde{G}^*$ be non-singular, where $G$ and $\tilde{G}$ are defined as in (6.20) and (6.23), respectively. If $\|A^{-1/2}\delta AA^{-1/2}\| < 1$ and $\|\delta LL^{-1}\|_F < 1$, then*

$$\begin{aligned} \sqrt{\sum_{i=1}^{r}\left(\frac{|\lambda_i - \tilde{\lambda}_{\tau(i)}|}{\sqrt{|\tilde{\lambda}_{\tau(i)}|}\max\{\sqrt{|\lambda_i|}, \sqrt{|\lambda_{\tau(i)}|}\}}\right)^2} &\leq \frac{2\,\|F\|\,\|\tilde{F}\|}{1 - \|\delta LL^{-1}\|}\cdot \\ \cdot\left[\frac{\|A^{-1/2}\delta AA^{-1/2}\|_F}{\sqrt{1 - \|A^{-1/2}\delta AA^{-1/2}\|}} + \|F\|\,\|\tilde{F}\|\frac{\|L^{-*}\delta ML^{-1}\|_F}{1 - \|\delta LL^{-1}\|}\right]&, \end{aligned} \tag{6.25}$$

*for some permutation of indices $\tau(i) \in \{1, \ldots, r\}$. Here $F$ and $\tilde{F}$ are $J$-unitary matrices which simultaneously diagonalize the pairs $(L^{-*}D^*ADL^{-1}, J)$ and*

$$((I + \delta LL^{-1})^{-*}L^{-*}D^*(A + \delta A)DL^{-1}(I + \delta LL^{-1})^{-1}, J),$$

*respectively.*

**Proof:** The assumption that $GM^{-1}G^*$ is non-singular implies the existence of a non-singular matrix $X = [\, X_1 \quad X_0 \,]$ which simultaneously diagonalizes the pair $(H, M)$ (this has been already described in (6.20) – (6.22)). Similarly, the assumption that $\tilde{G}\tilde{M}^{-1}\tilde{G}^*$ is non-singular, implies the existence of a non-singular matrix $\tilde{X} = [\, \tilde{X}_1 \quad \tilde{X}_0 \,]$ which simultaneously diagonalizes the pair $(\tilde{H}, \tilde{M})$, as in (6.24).

Before we continue with the proof, we define two auxiliary matrices, $S = A^{1/2}DL^{-1}$ and its perturbation

$$\tilde{S} = (I + A^{-1/2}\delta AA^{-1/2})^{1/2}A^{1/2}D\tilde{L}^{-1}.$$

Let

$$S = U\,[\,\Sigma \quad 0\,]\,F^{-1}, \qquad \tilde{S} = \tilde{U}\,[\,\tilde{\Sigma} \quad 0\,]\,\tilde{F}^{-1},$$

be HSVDs of the matrices $S$ and $\tilde{S}$, respectively. This implies that

$$A^{1/2}DX_{:,i} = \sqrt{|\lambda_i|}U_{:,i}, \quad j = 1, \ldots r, \tag{6.26}$$

and similarly for perturbed quantities, that is,

$$(I + A^{-1/2}\delta AA^{-1/2})^{1/2})A^{1/2}D\tilde{X}_{:,i} = \sqrt{|\tilde{\lambda}_i|}\tilde{U}_{:,i}, \quad j = 1, \ldots r, \tag{6.27}$$

where $X = L^{-1}F$ and $\tilde{X} = \tilde{L}^{-1}\tilde{F}$, and $X_{:,i}$ denotes the $i$-th column of $X$. From (6.24) it follows that

$$(H + \delta H)\tilde{X}_1 = (M + \delta M)\tilde{X}_1\tilde{\Lambda}_1,$$

where $\tilde{\Lambda}_1 = \tilde{\Sigma}^2 J_1 = \operatorname{diag}(\tilde{\lambda}_1, \ldots, \tilde{\lambda}_r)$, and similarly for unperturbed quantities. Now, by multiplying the above equality by $X_1$ from the left, and rearranging, we get the following component-wise inequality

$$|\lambda_i - \tilde{\lambda}_j||(X_1^*)_{:,i}M(\tilde{X}_1)_{:,j}| \leq |(X_1^*)_{:,i}D^*\delta AD(\tilde{X}_1)_{:,j}| + |(X_1^*)_{:,i}\delta M(\tilde{X}_1)_{:,j}||\tilde{\lambda}_j|,$$

for all $(i, j)$ where $i, j = 1, \ldots r$. From this, using (6.26) and (6.27), it follows

$$|\lambda_i - \tilde{\lambda}_j||(X_1)_{:,i}M(\tilde{X}_1)_{:,j}| \;\leq$$

$$\sqrt{|\lambda_i|}\sqrt{|\tilde{\lambda}_j|}|U_{:,i}^* A^{-1/2}\delta A A^{-1/2}(I + A^{-1/2}\delta A A^{-1/2})^{-1/2})\tilde{U}_{:,j}|$$
$$+|(F_1^*)_{:,i} L^{-*}\delta M \tilde{L}^{-1}(\tilde{F}_1)_{:,j}||\tilde{\lambda}_j|.$$

Squaring each term, and adding them all together, we obtain

$$\sqrt{\sum_{i=1}^{r}\left(\frac{|\lambda_i - \tilde{\lambda}_j|}{\sqrt{|\tilde{\lambda}_j|}\max\{\sqrt{|\lambda_i|},\sqrt{|\lambda_j|}\}}\cdot|(X_1^*)_{:,i}M(\tilde{X}_1)_{:,j}|^2\right)^2} \leq$$
$$2\,\|A^{-1/2}\delta A A^{-1/2}(I + A^{-1/2}\delta A A^{-1/2})^{-1/2})\|_F + 2\,\|F\|\,\|\tilde{F}\|\|L^{-*}\delta M \tilde{L}^{-1}\|_F.$$

By applying Lemma 3.3, and using the fact that

$$\frac{1}{\sigma_{\min}(X_1^* M \tilde{X}_1)} \leq \frac{1}{\sigma_{\min}(X^* M \tilde{X})}$$

it follows that there exists a permutation of $\{1, \ldots, r\}$, such that

$$\sqrt{\sum_{i=1}^{r}\left(\frac{|\lambda_i - \tilde{\lambda}_{\tau(i)}|}{\sqrt{|\tilde{\lambda}_{\tau(i)}|}\max\{\sqrt{|\lambda_i|},\sqrt{|\lambda_{\tau(i)}|}\}}\right)^2} \leq 2\,\|X^* M \tilde{X}\|\cdot$$
$$\cdot\left[\frac{\|A^{-1/2}\delta A A^{-1/2}\|_F}{\sqrt{1 - \|A^{-1/2}\delta A A^{-1/2}\|}} + \|F\|\,\|\tilde{F}\|\frac{\|L^{-*}\delta M L^{-1}\|_F}{1 - \|\delta L L^{-1}\|}\right]. \quad (6.28)$$

In (6.28) we have used the inequalities

$$\|A^{-1/2}\delta A A^{-1/2}(I + A^{-1/2}\delta A A^{-1/2})^{-1/2})\|_F \leq \frac{\|A^{-1/2}\delta A A^{-1/2}\|_F}{\sqrt{1 - \|A^{-1/2}\delta A A^{-1/2}\|}},$$

and

$$\|L^{-*}\delta M \tilde{L}^{-1}\|_F = \|L^{-*}\delta M L^{-1}(I + \delta L L^{-1})^{-1}\|_F \leq \frac{\|L^{-*}\delta M L^{-1}\|_F}{1 - \|\delta L L^{-1}\|}.$$

Note that,

$$\|X^* M \tilde{X}\| = \|X^* L^* J L \tilde{L}^{-1}\tilde{L}\tilde{X} \leq \|F\|\|\tilde{F}\|\|I + \delta L L^{-1}\|.$$

Now, by inserting the above inequality in (6.28) we obtain (6.25). ■

Note that the bound (6.25) contains $\|F\|$ and $\|\tilde{F}\|$, later on we will show how one can bound $\|\tilde{F}\|$ in the term of $\|F\|$, but first we state the second theorem.

Let $(H, M)$ and $(\tilde{H}, \tilde{M}) = (H + \delta H, M + \delta M)$ be Hermitian pairs defined by

$$H = D^* A D, \quad M = L^* J L, \quad \tilde{H} = D^*(A + \delta A) D, \quad \tilde{M} = (L + \delta L)^* J (L + \delta L), \tag{6.29}$$

where $D$ is non-singular, $A$, $A + \delta A$ are positive definite and $M$, $\tilde{M}$ are non-singular. The following theorem contains relative variant of Weyl-type of bound. This bound is obtained by using result (1.32) from Chapter 1.

By slightly modifying the proof of [9, Theorem 5.2.1], we can derive theorem for matrix pairs.

**Theorem 6.5** *Let $(H, M)$ and $(\tilde{H}, \tilde{M})$ be defined as in (6.29) and let all other matrices be defined as in Theorem 6.4. Let $W = \begin{bmatrix} w_1 & \dots & w_n \end{bmatrix}$ be $J$-unitary matrix which simultaneously diagonalizes the pair $(LD^{-1}A^{-1}D^{-*}L^*, J)$. If $\|A^{-1/2}\delta A A^{-1/2}\| < 1$ and if*

$$\max_i \kappa_{rel}(\lambda_i)\|\delta L L^{-1}\| < \frac{1}{2}, \quad \text{where} \quad \kappa_{rel}(\lambda_i) = |w_i^* w_i|,$$

*then*

$$\frac{|\lambda_i - \tilde{\lambda}_i|}{|\lambda_i|} \leq \frac{a + 2b}{(1-a)(1-2b)}, \tag{6.30}$$

*where*

$$a = \|A^{-1/2}\delta A A^{-1/2}\| \quad \text{and} \quad b = \|\delta L L^{-1}\|\, \kappa_{rel}(\lambda_i). \tag{6.31}$$

**Proof**. Using (6.20) and (6.21) and the fact that $M = L^* J L$ we can write the eigenvalue problem (6.1) in equivalent form as ordinary eigenvalue problem

$$G_{new} J G_{new}^* y \;=\; \mu y, \tag{6.32}$$

where $G_{new} = A^{-1/2} D^{-*} L^*$, $y = A^{1/2} D x$ and $\mu = 1/\lambda$. Since

$$A + \delta A = A^{1/2}(I + A^{-1/2}\delta A A^{-1/2})A^{1/2}, \quad \text{and} \quad (L + \delta L) = (I + \delta L L^{-1})L,$$

the corresponding perturbed problem has the following form

$$D_1 G_{new} D_2 J D_1^* G_{new}^* D_2 \tilde{y} \;=\; \tilde{\mu}\tilde{y}, \tag{6.33}$$

where $D_1 = (I + A^{-1/2}\delta A A^{-1/2})^{-1/2}$ and $D_2 = I + \delta L L^{-1}$. Using the bound (1.32) and the fact that $\lambda$ and $\mu$ have the same eigenvectors, that is, $\kappa_{rel}(\mu_i) = \kappa_{rel}(\lambda_i)$, we obtain the following bound for $\tilde{\mu}/\mu$,

$$(1-a)\,(1-2b) \quad \leq \frac{\tilde{\mu}}{\mu} \leq \quad \frac{1+2b}{1-a}, \tag{6.34}$$

where $a$ and $b$ are defined as in (6.31). Here we have used

$$\|D_1\|^2 \leq \frac{1}{1-\|A^{-1/2}\delta A A^{-1/2}\|\|} = \frac{1}{1-a} \quad \text{and} \quad \|I - D_2\| = \|\delta L L^{-1}\| = b.$$

Now by simple manipulation from (6.34) follows

$$\frac{1-a}{1+2b} \quad \leq \frac{\tilde{\lambda}}{\lambda} \leq \quad \frac{1}{(1-a)\,(1-2b)},$$

from which follows (6.30). ■

## 6.2 Eigensubspace perturbation bounds

The second result contains a perturbation bound for the sines of the canonical angles between eigenspaces of Hermitian pair $(H, M)$ from (6.18) and its perturbation $(\tilde{H}, \tilde{M}) = (H + \delta H, M + \delta M)$ which is defined by (6.19).

Again we assume that there exist non-singular matrices $X$ and $\tilde{X}$ which simultaneously diagonalize the pairs $(H, M)$ and $(\tilde{H}, \tilde{M}) = (H + \delta H, M + \delta M)$ as in (6.22) and (6.24), respectively. But here, we use slightly different decomposition of matrices $X$ and $\tilde{X}$.

We write $X$ and $\tilde{X}$ as

$$X = [\,X_{11} \quad X_{12} \quad X_0\,] \qquad \tilde{X} = [\,\tilde{X}_{11} \quad \tilde{X}_{12} \quad \tilde{X}_0\,],$$

where $X_{11}, \tilde{X}_{11} \in \mathbf{C}^{n\times k}$ and $X_{12}, \tilde{X}_{12} \in \mathbf{C}^{n\times r-k}$. Thus, (6.22) can be written as

$$\begin{aligned}
\begin{bmatrix} X_{11}^* \\ X_{12}^* \\ X_0^* \end{bmatrix} H\,[\,X_{11} \quad X_{12} \quad X_0\,] &= \begin{bmatrix} \Sigma_{11}^2 & 0 & 0 \\ 0 & \Sigma_{22}^2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \equiv \begin{bmatrix} |\Lambda_{11}| & 0 & 0 \\ 0 & |\Lambda_{22}| & 0 \\ 0 & 0 & 0 \end{bmatrix}, \\
\begin{bmatrix} X_{11}^* \\ X_{12}^* \\ X_0^* \end{bmatrix} M\,[\,X_{11} \quad X_{12} \quad X_0\,] &= \begin{bmatrix} J_{11} & 0 & 0 \\ 0 & J_{22} & 0 \\ 0 & 0 & J_2 \end{bmatrix} = \widehat{J},
\end{aligned} \tag{6.35}$$

where $\widehat{J}$ is diagonal and possibly a permutation of $J$. Similarly, for perturbed quantities, (6.24) can be written as

$$\begin{aligned}
\begin{bmatrix} \tilde{X}_{11}^* \\ \tilde{X}_{12}^* \\ \tilde{X}_0^* \end{bmatrix} \tilde{H} \, [\, \tilde{X}_{11} \quad \tilde{X}_{12} \quad \tilde{X}_0 \,] &= \begin{bmatrix} \tilde{\Sigma}_{11}^2 & 0 & 0 \\ 0 & \tilde{\Sigma}_{22}^2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \equiv \begin{bmatrix} |\tilde{\Lambda}_{11}| & 0 & 0 \\ 0 & |\tilde{\Lambda}_{22}| & 0 \\ 0 & 0 & 0 \end{bmatrix}, \\
\begin{bmatrix} \tilde{X}_{11}^* \\ \tilde{X}_{12}^* \\ \tilde{X}_0^* \end{bmatrix} M \, [\, \tilde{X}_{11} \quad \tilde{X}_{12} \quad \tilde{X}_0 \,] &= \begin{bmatrix} J_{11} & 0 & 0 \\ 0 & J_{22} & 0 \\ 0 & 0 & J_2 \end{bmatrix} = \widehat{J}.
\end{aligned} \tag{6.36}$$

As already mentioned, our aim is to derive a bound for the sines of the canonical angles between eigenspaces $\mathcal{X}_{11} = \mathcal{R}(X_{11})$ and $\tilde{\mathcal{X}}_{11} = \mathcal{R}(\tilde{X}_{11})$. Before we state our theorem we will present two kinds of relative gaps between eigenvalues from the sets $\rho_{n-k} = \{\lambda_{k+1}, \ldots, \lambda_n\}$, and $\tilde{\rho}_k = \{\tilde{\lambda}_1, \ldots, \tilde{\lambda}_k\}$. Let

$$Relgap = \min_{\substack{i \in \rho_{n-k} \\ j \in \tilde{\rho}_k}} \frac{|\lambda_i - \tilde{\lambda}_j|}{\sqrt{|\lambda_i||\tilde{\lambda}_j|}} \qquad rg = \min_{\substack{i \in \rho_{n-k} \\ j \in \tilde{\rho}_k}} \frac{|\lambda_i - \tilde{\lambda}_j|}{|\tilde{\lambda}_j|}. \tag{6.37}$$

Now we can state the second theorem.

**Theorem 6.6** *Let $(H, M)$ be a Hermitian pair defined by (6.18) and let $(\tilde{H}, \tilde{M})$ be perturbed pair defined by (6.19). Let $X = [\, X_{11} \quad X_{12} \quad X_0 \,]$ and $\tilde{X} = [\, \tilde{X}_{11} \quad X_{12} \quad \tilde{X}_0 \,]$, be non-singular matrices which simultaneously diagonalize the pairs $(H, M)$ and $(\tilde{H}, \tilde{M})$, as in (6.35) and (6.36), respectively. If $\|A^{-1/2}\delta A A^{-1/2}\| < 1$ and $\|\delta L L^{-1}\| < 1$, then*

$$\begin{aligned}
\|\sin\Theta(\mathcal{X}_{11}, \tilde{\mathcal{X}}_{11})\|_F \;&\leq\; 2\,\|F\|\|\tilde{F}\|\|L\|^2(1 + \|\delta L L^{-1}\|) \cdot \\
&\cdot \left( \frac{\Psi}{Relgap} + \frac{\|F\|\,\|\tilde{F}\|}{rg} \frac{\|L^{-*}\delta M L^{-1}\|_F}{1 - \|\delta L L^{-1}\|} \right),
\end{aligned} \tag{6.38}$$

*where $\sin\Theta(\mathcal{X}_{11}, \tilde{\mathcal{X}}_{11})$ is diagonal matrix with sines of canonical angles between $\mathcal{X}_{11}$ and $\tilde{\mathcal{X}}_{11}$ on its diagonal,*

$$\Psi = \frac{\|A^{-1/2}\delta A A^{-1/2}\|_F}{\sqrt{1 - \|A^{-1/2}\delta A A^{-1/2}\|}},$$

*and $F$ and $\tilde{F}$ are defined as in Theorem 6.4.*

**Proof.** If we write $X_2 = [\, X_{22} \quad X_0 \,]$, then $X = [\, X_{11} \quad X_2 \,]$. Similarly, if we write $\tilde{X}_2 = [\, \tilde{X}_{22} \quad \tilde{X}_0 \,]$, then $\tilde{X} = [\, \tilde{X}_{11} \quad \tilde{X}_2 \,]$. Denote the inverse of $X$ by

$$X^{-1} \equiv V = \begin{bmatrix} V_{11}^* \\ V_2^* \end{bmatrix}.$$

Further, let $\tilde{X}_{11} = \tilde{Q}_{11}\tilde{R}_{11}$ and $V_2 = Q_2R_2$ be QR decompositions of the matrices $\tilde{X}_{11}$ and $V_2$, respectively, where $\tilde{R}_{11}$ and $R_2$ are non-singular. Since $\|\sin\Theta(\mathcal{X}_{11},\tilde{\mathcal{X}}_{11})\|_F = \|Q_2^*\tilde{Q}_{11}\|_F$, we can write

$$\begin{aligned} \|\sin\Theta(\mathcal{X}_{11},\tilde{\mathcal{X}}_{11})\|_F &= \|R_2^{-*}V_2^*\tilde{X}_{11}\tilde{R}_{11}^{-1}\|_F \\ &\leq \|R_2^{-1}\|\|\tilde{R}_{11}^{-1}\|\|V_2^*\tilde{X}_{11}\|_F \\ &\leq \|X^{-1}\|\|\tilde{X}^{-1}\|\|X_2^*M\tilde{X}_{11}\|_F. \end{aligned} \tag{6.39}$$

Here we have used the fact that

$$V_2^* = \begin{bmatrix} J_{22} & 0 \\ 0 & J_2 \end{bmatrix} X_2^*M$$

and that

$$\|R_2^{-1}\| \leq \|X^{-1}\|, \quad \|\tilde{R}_1^{-1}\| \leq \|\tilde{X}^{-1}\|.$$

Now we can proceed, by deriving a bound for $\|X_2^*M\tilde{X}_{11}\|_F$. Denote $\Lambda_2$ as

$$\Lambda_2 = \begin{bmatrix} \Lambda_{22} & 0 \\ 0 & 0 \end{bmatrix}.$$

Similarly as in the proof of Theorem 6.4, we can write

$$\Lambda_2X_2^*M\tilde{X}_{11} - X_2^*M\tilde{X}_{11}\tilde{\Lambda}_{11} = -X_2^*\delta H\tilde{X}_1 + X_2^*\delta M\tilde{X}_1\tilde{\Lambda}_{11}.$$

It is easy to show, that for all pairs of indices $(i,j)$ where $i \in \rho_{n-k}$ and $j \in \rho_k$, we have

$$|(X_2^*M\tilde{X}_{11})_{ij}||(\Lambda_2)_{ii} - (\tilde{\Lambda}_{11})_{jj}| \leq |X_2^*D^*\delta AD\tilde{X}_{11}|_{ij} + |X_2^*\delta M\tilde{X}_{11}|_{ij}|\tilde{\Lambda}_{jj}|.$$

Squaring each term, adding them all together, and using (6.26) and (6.27), we obtain

$$\begin{aligned} \|X_2^*M\tilde{X}_{11}\|_F &\leq \frac{2}{\min\limits_{\substack{i\in\rho_{n-k}\\ j\in\tilde{\rho}_k}} \frac{|\lambda_i-\tilde{\lambda}_j|}{\sqrt{|\lambda_i||\tilde{\lambda}_j|}}}\|A^{-1/2}\delta AA^{-1/2}(I + A^{-1/2}\delta AA^{-1/2})^{-1/2})\|_F \\ &+ \frac{2}{\min\limits_{\substack{i\in\rho_{n-k}\\ j\in\tilde{\rho}_k}} \frac{|\lambda_i-\tilde{\lambda}_j|}{|\tilde{\lambda}_j|}}\|X^*\delta M\tilde{X}\|_F. \end{aligned}$$

In the above inequality we used the fact that $(a+b)^2 \leq 2(a^2+b^2)$. Further, it follows that

$$\|X_2^*M\tilde{X}_{11}\|_F \leq \frac{2}{Relgap}\cdot\frac{\|A^{-1/2}\delta A\tilde{A}^{-1/2}\|_F}{\sqrt{1-\|A^{-1/2}\delta A\tilde{A}^{-1/2}\|}} + \frac{2}{rg}\|F\|\|\tilde{F}\|\frac{\|L^{-*}\delta ML^{-1}\|_F}{1-\|\delta LL^{-1}\|}.$$

Now, the bound (6.38) follows by inserting the above inequality in (6.39). ■

Finally, we show how one can bound $\|\tilde{F}\|$ in terms of $\|F\|$ in Theorems 6.4 and 6.6. Matrices $F$ and $\tilde{F}$ are $J$-unitary and simultaneously diagonalize the pairs $(L^{-*}D^*ADL^{-1}, J)$ and

$$((I+\delta LL^{-1})^{-*}L^{-*}D^*(A+\delta A)DL^{-1}(I+\delta LL^{-1})^{-1}, J),$$

respectively. Note that $F$ is obtained by HSVD of the matrix $S = A^{1/2}DL^{-1}$ and $\tilde{F}$ is obtained by HSVD of

$$\tilde{S} = (I + A^{-1/2}\delta AA^{-1/2})^{1/2}S(I+\delta LL^{-1})^{-1}.$$

We divide our procedure in two steps. In the first step we derive a bound for $\|F_p\|$, where $F_p$ is auxiliary matrix obtained by HSVD of the matrix $S_p = (I+A^{-1/2}\delta AA^{-1/2})^{1/2}S$. Note that $S_p$ is obtained by multiplicative perturbation of $S$ from the left. Let

$$S_p \equiv (I + A^{-1/2}\delta AA^{-1/2})^{1/2}S = U_p \, [\, \Sigma_p \quad 0\,]\, F_p^{-1},$$

be an HSVD of $S_p$. Note, that this is exactly the same perturbation of the matrix $S$ as the perturbation of the matrix $G$ from the proof of Theorem 5.2. Thus, we can use (5.29) to bound $\|F_p\|$ in terms of $\|F\|$, that is, from (5.29) it follows that

$$\|F^*JF_p\| \le \left(\frac{\Psi}{2} + \sqrt{1+\frac{\Psi^2}{4}}\right) = \Psi_0, \tag{6.40}$$

where

$$\Psi = \frac{\|A^{-1/2}\delta AA^{-1/2}\|_F}{\sqrt{1-\|A^{-1/2}\delta AA^{-1/2}\|}}, \quad \text{and} \quad \Psi_0 = \frac{\Psi}{2} + \sqrt{1+\frac{\Psi^2}{4}}.$$

Since,

$$\|F^{-*}\|^{-1}\|F_p\| \le \|F^*JF_p\| \quad \text{and} \quad \|F\|\|F_p^{-1}\|^{-1} \le \|F^*JF_p\|,$$

and $\|F^{-*}\| = \|F\|$, $\|F_p^{-1}\| = \|F_p\|$, we have the following two bounds

$$\frac{\|F\|}{\Psi_0} \le \|F_p\| \le \|F\| \cdot \Psi_0. \tag{6.41}$$

In the second step, first note that $\tilde{S} = S_p(I+\delta LL^{-1})^{-1}$, which means that $\tilde{S}$ is obtained by multiplicative perturbation of the matrix $S_p$ from the right. If we write

$$(I+\delta LL^{-1})^{-1} = I + E,$$

then $\tilde{S}$ can be written as $\tilde{S} = S_p(I+E)$, which is exactly the same perturbation of the matrix $S_p$ as the perturbation of the matrix $G$ which is discussed after Theorem 5.4. Thus, from (5.55) it follows that

$$\|\tilde{F}\| \le \frac{\|F_p\|}{\sqrt{1-2\varrho\|F_p\|^2}}, \tag{6.42}$$

where

$$\varrho \equiv \frac{3\|E\|_F - 4\|E\|^2}{1-3\|E\|+2\|E\|^2} < \frac{1}{2\|F_p\|^2}.$$

If we additionally assume that

$$\|\delta L L^{-1}\|_F \le \frac{1}{4+6\|F_p\|^2}, \tag{6.43}$$

then it follows that

$$\|E\|_F \le \frac{\|\delta L L^{-1}\|_F}{1-\|\delta L L^{-1}\|} < \frac{1}{3}\cdot\frac{1}{1+2\|F_p\|^2},$$

which further implies

$$\varrho \le \frac{3\|E\|_F}{1-3\|E\|_F} < \frac{1}{2\|F_p\|^2}.$$

Now, from (6.42) and (6.41) we have

$$\|\tilde{F}\| \le \frac{\Psi_0\|F\|}{\sqrt{1-2\varrho\Psi_0^2\|F\|^2}}. \tag{6.44}$$

Note that by using (6.41), we can write (6.43) as

$$\|\delta L L^{-1}\|_F \le \frac{1}{4+6\dfrac{\|F\|^2}{\Psi_0^2}}. \tag{6.45}$$

Thus, with additional condition (6.45), we can rewrite Theorem 6.4 and Theorem 6.6 so that in the bounds (6.25) and (6.38) only the norm of the unperturbed $J$-unitary matrix $F$ appears.

As an illustration we present a corollary of Theorem 6.6, whose bound is composed only of unperturbed quantities.

**Corollary 6.3** *Let $(H,M)$ be Hermitian pair defined by (6.18) and let $(\tilde{H},\tilde{M})$ be perturbed pair defined by (6.19). Let $X = [\,X_{11} \quad X_{12} \quad X_0\,]$ and $\tilde{X} = [\,\tilde{X}_{11} \quad X_{12} \quad \tilde{X}_0\,]$ be non-singular matrices which simultaneously diagonalize the pairs $(H,M)$ and $(\tilde{H},\tilde{M})$, as in (6.35) and (6.36), respectively. If $\|A^{-1/2}\delta A A^{-1/2}\| < 1$ and*

$$\|\delta L L^{-1}\| < \frac{\Psi_0^2}{4\Psi_0^2 + 6\|F\|^2},$$

*then*

$$\begin{aligned}\|\sin\Theta(\mathcal{X}_{11},\tilde{\mathcal{X}}_{11})\|_F \;\leq\;& \frac{2\,\Psi_0\|F\|^2}{\sqrt{1-2\varrho\Psi_0^2\|F\|^2}}\|L\|^2(1+\|\delta L L^{-1}\|)\\ &\cdot\left(\frac{\Psi}{Relgap}+\frac{1}{rg}\,\frac{\Psi_0\|F\|}{\sqrt{1-2\varrho\Psi_0^2\|F\|^2}}\,\frac{\|L^{-*}\delta M L^{-1}\|_F}{1-\|\delta L L^{-1}\|}\right).\end{aligned}$$

*Here $\sin\Theta(\mathcal{X}_{11},\tilde{\mathcal{X}}_{11})$ is diagonal matrix with sines of canonical angles between $\mathcal{X}_{11}$ and $\tilde{\mathcal{X}}_{11}$, on its diagonal,*

$$\Psi = \frac{\|A^{-1/2}\delta A A^{-1/2}\|_F}{\sqrt{1-\|A^{-1/2}\delta A A^{-1/2}\|}}, \qquad \Psi_0 = \frac{\Psi}{2}+\sqrt{1+\frac{\Psi^2}{4}},$$

*and $F$ is $J$-unitary matrix which diagonalizes the pair $(L^{-*}D^*ADL^{-1}, J)$.*

**Proof.** The proof follows by inserting (6.44) i to the (6.38). ■

The following numerical example illustrates the quality of bounds described in this chapter. We will assume that all matrices of the pairs $(H,M)$ and $(\tilde{H},\tilde{M})$, are non-singular. This allows us to include Theorem 6.5 as well.

**Example 6.1** *Let $H = D^*AD$, where*

$$A = \begin{bmatrix} 1.383 & 0.937 & 0.878 & 0.675\\ 0.937 & 1.33 & 0.610 & 0.472\\ 0.878 & 0.610 & 1.347 & 1.014\\ 0.675 & 0.472 & 1.014 & 1.150\end{bmatrix}$$

*and $D = \mathrm{diag}\,(10^6, 10, 0.1, 10^{-4})$. Let $M = L^*JL$ where*

$$L = \begin{bmatrix} 0.680 & 0 & 0 & 0\\ -0.526 & 0.522 & 0 & 0\\ -0.049 & 0.245 & 0.766 & 0\\ 0.783 & 0.559 & 0.559 & 0.894\end{bmatrix}.$$

*Further, let* $\tilde{H} = D^*(A + \delta A)D$ *and* $\tilde{M} = (L + \delta L)^* J(L + \delta L)$ *be corresponding perturbed matrices, where*

$$\delta A = \begin{bmatrix} 1.850 \cdot 10^{-5} & 2.706 \cdot 10^{-5} & 2.786 \cdot 10^{-5} & 4.431 \cdot 10^{-5} \\ 2.706 \cdot 10^{-5} & 7.650 \cdot 10^{-5} & 2.3318 \cdot 10^{-5} & 3.226 \cdot 10^{-6} \\ 2.786 \cdot 10^{-5} & 2.332 \cdot 10^{-5} & 1.733 \cdot 10^{-6} & 2.125 \cdot 10^{-5} \\ 4.431 \cdot 10^{-5} & 3.226 \cdot 10^{-6} & 2.125 \cdot 10^{-5} & 7.042 \cdot 10^{-5} \end{bmatrix},$$

*and*

$$\delta L = \begin{bmatrix} 3.897 \cdot 10^{-5} & 0 & 0 & 0 \\ 5.976 \cdot 10^{-7} & 2166 \cdot 10^{-5} & 0 & 0 \\ 2.273 \cdot 10^{-5} & 2.765 \cdot 10^{-5} & 6.316 \cdot 10^{-6} & 0 \\ -1.711 \cdot 10^{-6} & 8.638 \cdot 10^{-6} & -1.891 \cdot 10^{-6} & 3.022 \cdot 10^{-5} \end{bmatrix}.$$

*The eigenvalues of the pair* $(H, M)$ *are (properly rounded)*

$$\lambda_1 = 2.995, \quad \lambda_2 = 1.342 \cdot 10^{-2}, \quad \lambda_3 = 4.841 \cdot 10^{-9}, \quad \lambda_4 = -2.552 \cdot 10^2.$$

*Here*

$$\max_i \frac{|\tilde{\lambda}_i - \lambda_i|}{|\lambda_i|} \approx 1.0132 \cdot 10^{-4},$$

*and the left hand side of (6.25) (the exact perturbation) is equal to* $1.185 \cdot 10^{-4}$.

*Since* $\|\delta H\| = 1.850 \cdot 10^7$ *and* $\|\delta M\| = 9.588 \cdot 10^{-5}$ *and* $\gamma = 0.589$, *the classical bounds (6.4) and (6.6) are useless.*

*Theorem 6.3 and Corollary 6.2 give*

$$\max_i \frac{|\tilde{\lambda}_i - \lambda_i|}{|\lambda_i|} \leq 2.3414 \cdot 10^{-4}.$$

*The right hand side in the bound (6.25) of Theorem 6.4 is* $6.874 \cdot 10^{-4}$ *and the bound (6.30) of Theorem 6.5 gives*

$$\max_i \frac{|\tilde{\lambda}_i - \lambda_i|}{|\lambda_i|} \leq 3.956 \cdot 10^{-4},$$

*where* $\max_i \kappa_{rel}(\lambda_i) \approx 1$.

*Let us now consider invariant subspaces. The bound (6.38) of Theorem 6.6 bounds* $\sin \Theta_i = \sin \Theta(x_i, \tilde{x}_i)$, *where* $x_i$ *is the eigenvector of* $\lambda_i$.

*Table 6.1 gives the corresponding relative gaps from (6.37), the exact perturbations and the bound (6.38).*

| $i$ | *1* | *2* | *3* | *4* |
|---|---|---|---|---|
| $RelGap(\tilde{\lambda}_i, \Lambda_2)$ | $1.083 \cdot 10^5$ | $1.379 \cdot 10^2$ | $01.379 \cdot 10^2$ | $1.666 \cdot 10^3$ |
| $rg(\tilde{\lambda}_i, \Lambda_2)$ | 1 | 0.995 | 0.995 | 1 |
| $\\|\sin \Theta_i\\|_F$ | $4.239 \cdot 10^{-5}$ | $3.679 \cdot 10^{-5}$ | $1.666 \cdot 10^{-7}$ | $6.752 \cdot 10^{-9}$ |
| *Theorem 6.6* | $4.412 \cdot 10^{-4}$ | $4.453 \cdot 10^{-4}$ | $4.435 \cdot 10^{-4}$ | $1.961 \cdot 10^{-7}$ |

Table 6.1: Eigenvector perturbations

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## Sažetak

U ovoj radnji proučavamo *relativnu perturbacijsku teoriju* za nekoliko različitih vrsta dekompozicija na svojstvene vrijednosti. Proučavamo sljedeće svojstvene probleme: hermitski problem svojstvenih vrijednosti, hiperbolini problem svojstvenih vrijednosti i generalitizrani problem svojstvenih vrijednosti.

U uvodu objašnjavamo zašto je zanimljiva relativna perturbacijska teorija i dajemo pregled postojećih perturbacijskih rezultata klasične i relativne perturbacijske teorije, te pregled naših rezultata.

U poglavlju 2 predstavljamo CS-dekompoziciju $J$ unitarne matrice, gdje je $J$ dijagonalna matrica predznaka. Navodimo neke primjene te dekompozicije na $J$ unitarne matrice, uključujući i perturbacijsku ocjenu za $J$ unitarne matrice. Rezultati ovog poglavlja su primjenjeni u sljedećim poglavljima.

U poglavlju 3 predstavljamo relativnu verziju perturbacijske ocjene za svojstvene vrijednosti Hoffman-Wielandtovg tipa za hermitske matrice i relativne verziju poznatog $\sin\Theta$ teorema Davisa i Kahana za jednostavne invarijantne potprostore. Nadalje, poglavlje 4 sadrži relativnu linearnu i kvadratnu rezidualnu ocjenu za svojstvene vrijednosti hermitske matrice.

U poglavlju 5 predstavljamo relativnu verziju perturbacijske ocjene za svojstvene vrijednosti Hoffman-Wielandtovg tipa i relativne verziju poznatog $\sin\Theta$ teorema Davisa i Kahana za jednostavne invarijantne potprostore za hiperbolni problem svojstvenih vrijednosti $Hx = \lambda Jx$.

Zaključno, u poglavlju 6 predstavljamo relativnu perturbacijsku ocjenu za svojstvene vrijednosti i perturbacijsku ocjenu za svojstvene potprostore generaliziranog svojstvenog problema.

## Summary

In this thesis we consider the *relative perturbation theory* for various eigenvalue decompositions. We consider following eigenvalue problems: the Hermitian eigenvalue problem, the hyperbolic eigenvalue problem, and the generalized eigenvalue problem.

In the introduction we explain why are relative perturbation bounds of interest and state some of the existing classical and relative perturbation results, and give overview of our results.

In Chapter 2 we present CS-decomposition of a $J$-unitary matrix, where $J$ is a diagonal matrix of signs. We give some application of this decomposition to $J$-unitary matrices, including perturbation bounds for $J$-unitary matrices. The results of this chapter are applied in the subsequent chapters.

In Chapter 3 we present the relative version of the Hoffman-Wielandt-type bound for eigenvalues of Hermitian matrices, and relative variant of the well-known $\sin\Theta$ theorem by Davis and Kahan for simple invariant subspaces. Further, Chapter 4 contains relative linear and quadratic residual bounds for the Hermitian eigenvalue problem.

In Chapter 5 we present the relative version of Hoffman-Wielandt-type bound for the eigenvalues, and relative variant of $\sin\Theta$ theorem for simple invariant subspaces for the hyperbolic eigenvalue problem $Hx = \lambda Jx$.

Finally, in Chapter 6 we present relative eigenvalue perturbation bound and perturbation bound for eigensubspaces for the generalized eigenvalue problem.

# Životopis

Rođen sam 04.05.1963. godine u Osijeku. Do 1991. godine sam živio u Dardi, gdje sam završio osnovnu školu. U Osijeku sam 1981. godine završio srednju školu u tadašnjem CUO " Braća Ribar " na smjeru matematike i informatike, a nakon odsluženja vojnog roka 1982. nastavio sam studij u Osijeku na Pedagoškom Fakultetu.

Diplomirao sam 1987. godine i time stekao zvanje profesora matematike i fizike, te se 1988. godine zaposlio na Tekstilnom školskom centru u Osijeku. Te iste godine sam prešao na Građevinski fakultet u Osijeku kao asistent pripravnik. Poslijediplomski studij sam upisao 1989. u Novom Sadu na smjeru Numerička matematika, gdje sam do 1991. položio sve ispite i obranio sve seminarske radove koji su bili predviđeni planom i programom tog studija. Do završetka mi je ostala samo obrana magistarske radnje no poznata ratna zbivanja su me spriječila u tome. U kolovozu 1991. godine sam ostao bez svega što sam imao i svega što sam do tada napravio. Postajem prognanik i od tada živim kao podstanar u Osijeku.

Svršetkom godine 1992., upisujem se na drugu godinu posljediplomskog studija u Zagrebu na smjer numerička matematika i matematička informatika. Dana 12. 12. 1995. godine sam obranio magistarski rad pod naslovom "Perturbacije invarijantnih potprostora" na PMF-u u Zagrebu pred povjerenstvom u sastavu: prof. dr. sc. Vjeran Hari (predsjednik), prof. dr. sc. Branko Najman i doc. dr. sc. Ivan Slapničar (voditelj rada).

Nakon svršetka posljediplomskog studija, nastavio sam intenzivno surađivati s dr. sc. Ivanom Slapničarem. Rezultat naše dosadašnje suradnje su tri članka te nekoliko sudjelovanja na poznatim svjetskim skupovima.

Na poziv prof. dr. Ilse C. F. Ipsen, od 29. listopada do 1. studenog 1997. godine aktivno sam sudjelovao na *Sixth SIAM Conference on Applied Linear Algebra*, Snowbird, Utah, SAD. Nakon toga sam od 2. studenog do 20. prosinca boravio na The Pennsylvania State University, University Park, PA, kao gost prof. dr. Jesse L. Barlow-a. Tokom posijeta veći sam dio vremena posvetio proučavanju svojstvenog problema para hermitskih matrica.

Zaposlen sam na Građevinskom fakultetu u Osijeku kao asistent na predmetima matematika 1. i 2. koji se predaju na prvoj godini studija.

Također sam radio na Pedagoškom fakultetu kao vanjski suradnik, na predmetu matematička analiza 3. i 4. kao asistent.

Trenutno radim kao asistent (Wissenschaftliche Mitarbeiter) na Fernuniversitätu u Hagenu, Fachbereich Mathematik, Lehrgebiet Mathematische Physik, Njemačka.

# Biography

I was born in Osijek in 1963 and lived in Darda until 1991, where I finished elementary school. In 1981 I finished high school in Osijek and after one year of army service I continued study in Osijek on Faculty of Education.

I graduated in 1987, and in 1988 I started to work as a teacher at the Textile High School in Osijek. In the same year I started to work at the Faculty of Civil Engineering in Osijek as the scientific assistant. In 1989 I started post graduate studies in Novi Sad, speciality *Numerical Mathematics*. Until 1991 I completed all exams, but the well-known war events prevented me completing my master studies. In August 1991 I lost everything I had and became a displaced person. From then on I live in Osijek.

In 1992 I started the second year of post graduated studies in Zagreb on *Seminar for Numerical Mathematics and Mathematical Informatics*. I graduated in 1995 with M.S. thesis "Perturbations of Invariant Subspaces" at the Faculty of Mathematics and Natural Sciences, University of Zagreb, in front of the committee: Prof. Dr. Vjeran Hari (chair), Prof. Dr. Branko Najman and Prof. Dr. sc. Ivan Slapničar (supervisor).

After post graduate studies I continued to work with my Ph. D. thesis supervisor Dr. Ivan Slapničar. The result of our collaboration are three scientific articles and participation in several international congresses.

In 1997, upon invitation by Prof. Dr. Ilse C. F. Ipsen, I participated in *Sixth SIAM Conference on Applied Linear Algebra*, Snowbird, Utah, USA, and after I spent month and half as a scientific visitor of Prof. Dr. Jesse L. Barlow at the Pennsylvania State University, University Park, PA, USA.

Currently I am on scientific leave from the Faculty of Civil Engineering in Osijek, and I work at the Department of Mathematical Physics, Faculty of Mathematics, Fernuniversität in Hagen, Germany.