Guarding 1.5D Terrains with Demands

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Abstract

We consider the 1.5D terrain guarding problem in which every point on the terrain that is to be covered has an integer demand associated with it. The goal is to find a minimum cardinality set of guards such that each point is guarded by a number of guards satisfying its demand. We present an algorithm that yields a 6.7-approximation in the case where the minimum demand $d_{\min} < 5$, and a 3-approximation otherwise. To the best of our knowledge, this is the first constant factor approximation algorithm for this problem.

As in our previous result [6] we use a fractional solution to the linear programming relaxation of the corresponding covering problem to decide, for each point, the amount of demand that has to be satisfied from the left and right sides of the point.

1 Introduction

In the 1.5D terrain guarding problem we are given a polygonal region in the plane determined by an x-monotone polygonal chain, and the objective is to find the minimum number of guards to place on the chain such that every point in the polygonal region is guarded. This kind of guarding problems and its generalizations to 3-dimensions are motivated by optimal placement of antennas for communication networks (see [3, 1] and the references therein for more details).

The problem considered in this paper is generalization of the 1.5D terrain guarding problem. In the *1.5D terrain guarding problem with demands* we are given an x-monotone polygonal chain T in the plane, a set $G \subset T$ of guards and a set $N \subset T$ of points with the associated demand function $d_p : N \rightarrow \mathbb{Z}^+$. The goal is to find a minimum cardinality set of guards such that each point $p \in N$ is guarded by at least d_p different guards form this set.

One motivation for studying this version of the problem is that it allows for more robust guarding. Namely, none of the points should stay unguarded even if some of the guards collapse.

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Previous Work Chen et al. [3] claimed that the 1.5D-terrain guarding problem is NP-hard but they did not give a complete proof of the claim (see [5]). They also gave a linear time algorithm for *the left-guarding* problem, that is, the problem of placing the minimum number of guards on the chain such that each point of the chain is guarded from its left. Based on purely geometric arguments, Ben-Moshe et al. [1] gave the first constant-factor approximation algorithm for the 1.5D-terrain guarding problem, though they did not state the value of the approximation factor explicitly (it was claimed to be at least 6 in [8]). Clarkson and Varadarajan [4] gave constant factor approximation algorithms for a more general class of problems using ϵ-nets and showed that their technique can be used to get a constant approximation for the 1.5D-terrain guarding problem. King gave another geometric algorithm[1] with approximation factor 5.

Elbassioni et al. [6] presented a 4-approximation algorithm for the problem. Unlike most of the previous techniques, their method was based on rounding the linear programming relaxation of the corresponding covering problem. Besides the simplicity of the analysis, which mainly relies on decomposing the constraint matrix of the LP into totally balanced matrices, their algorithm generalizes to the weighted and partial versions of the problem.

Most recently, King and Krohn [11] resolved the question about the hardness of the problem and showed that problem is indeed NP-hard. Gibson et al. [7] obtained a PTAS for the standard 1.5D terrain guarding problem using a local search technique. Their analysis relied on a result by Mustafa and Ray [13].

Clarkson et al. [2] considered a number of geometric set covering problems with demands and gave, among other results, an LP-based algorithm that yields an approximation factor of $O(\log z^*)$ for set systems with bounded VC-dimension, where z^* is the value of the optimal solution of the standard covering LP relaxation. By a recent result of King [10], the set system arising in 1.5D-guarding has VC-dimension 4, implying an $O(\log z^*)$-approximation for the 1.5D-guarding problem with demands.

[1] King claimed that the the problem can be approximated within a factor of 4 in [9]; however, his analysis turned out to have an error that increases the approximation factor to 5.

Our Contribution We present an algorithm that yields a 6.7-approximation in the case where the minimum demand $d_{\min} < 5$, and a 3-approximation otherwise. The main idea of our approach is to compute the fractional solution of the corresponding LP and, based on this solution, to decide, for each point, the portion of demands which has to be met from the left side and that which has to be met from the right side. As a result, we end up with left and right guarding problems with demands, such that satisfying the portions of demands on the left and the right, we meet the original demand for each point.

The system matrix for the left and the right problems was shown in [6] to have a special structure; namely, the matrix is *totally balanced*. For that reason, it can be shown that the corresponding left and right guarding problems have integral optimal solutions as described in the book chapter by Kolen and Tamir [12], even in the case with demands. However, in order to get more insight into the problem we provide an alternative proof by constructing a very simple and easy to understand procedure that finds the optimal solution for the left and the right guarding problem. We show how to combine the left and right solutions to arrive at the claimed approximation guarantee.

2 Preliminaries

A terrain T is an x-monotone polygonal chain. Let V denote set of vertices of T, and $|V|$ is complexity of the terrain T. For two points $p, q \in T$ we say that p sees q and denote $p \sim q$ if the line segment connecting p and q does not go strictly below T. We say that p is seen from $S \subset T$ if there exists some $g \in S$ such that $p \sim g$. Let $N \subset T$, $|N| = n$, be some set of points with the demand function $d_p : N \to \mathbb{Z}^+$ defined. Let $G \subset T$, $|G| = m$, be some set of guards.

In the *1.5D terrain guarding problem with demands* the task is to find minimum set of guards $A \subseteq G$ such that every point p in N is guarded by at least d_p guards from A.

We write $p < q$ if point p is on the strict left of q. All approximation algorithms mentioned in the Previous Work part are based on the following *order claim*:

Lemma 1 *Let $a < b < c < d$ be four points on T. If $a \sim c$ and $b \sim d$, then $a \sim d$.*

For any point $p \in N$ we define $S(p)$ to be the set of guards from G that see p, $S_L(p)$ to be the set of guards from G that see point p strictly from the left and $S_R(p)$ the set of guards from G that see p strictly from the right.

3 Terrain guarding with demands

Consider the following integer LP formulation for the problem:

$$\text{minimize} \sum_{g \in G} x_g \qquad \text{(LP1)}$$

subject to

$$\sum_{g \in S(p)} x_g \geq d_p \qquad \forall p \in N \qquad (1)$$

$$x_g \in \{0,1\} \qquad \forall g \in G$$

Variable x_g indicates whether $g \in G$ is chosen as a guard and constraint (1) demands that every point $p \in N$ is guarded with at least d_p guards from G.

Let x^* denote the optimal solution to the LP relaxation.

Rounding large values. We fix some parameter $\alpha \in (0, 1/2)$ that will be defined later. We let $G_0 = \{g \in G : x^*_g \geq \alpha\}$, which we take into our final solution. Then we get a reduced problem by redefining $d'_p = d_p - |S(p) \cap G_0|$ for all $p \in N$, $N' = \{p \in N : d'_p \geq 1\}$, and $d_{\min} = \min\{d'_p : p \in N'\}$. Define further $G' = G \setminus G_0$, $S'(p) = S(p) \cap G'$ and similarly $S'_L(p)$ and $S'_R(p)$. Let

$$N_L = \{p \in N' \mid \sum_{g \in S'_L(p)} x^*_g \geq \frac{1}{2}(\sum_{g \in S'(p)} x^*_g - x^*_p)\}$$

$$N_R = \{p \in N' \mid \sum_{g \in S'_R(p)} x^*_g \geq \frac{1}{2}(\sum_{g \in S'(p)} x^*_g - x^*_p)\},$$

where we assume that $x^*_p = 0$ if $p \notin G'$.

For each $p \in N'$, we define

$$\begin{array}{rcl} d_{p,L} & = & \lceil \sum_{g \in S'_L(p)} x^*_g \rceil, \forall p \in N_L \\ d_{p,L} & = & \lfloor \sum_{g \in S'_L(p)} x^*_g + x^*_p \rfloor, \forall p \in N_R \\ d_{p,R} & = & \lceil \sum_{g \in S'_R(p)} x^*_g \rceil, \forall p \in N_R \\ d_{p,R} & = & \lfloor \sum_{g \in S'_R(p)} x^*_g + x^*_p \rfloor, \forall p \in N_L \end{array}$$

as the demand of point p that has to be satisfied from the guards that are to the left and right of p, respectively. Note that every point $p \in N'$ must be either in N_L or N_R.

Consider the LP formulation for the left-guarding problem:

$$\text{minimize} \sum_{g \in G'} x_g \qquad \text{(LP2)}$$

subject to

$$\sum_{g \in S'_L(p)} x_g \geq d_{p,L} \qquad \forall p \in N'$$

$$0 \leq x_g \leq 1 \qquad \forall g \in G'$$

and the corresponding dual:

$$\text{minimize} \sum_{p\in N'} d_{p,L} y_p - \sum_{g\in G'} z_g \qquad \text{(LP3)}$$

subject to

$$\sum_{p:\, g\in S'_L(p)} y_p - z_g \le 1 \qquad \forall g \in G'$$
$$y_p \ge 0 \qquad \forall p \in N'$$
$$z_g \ge 0 \qquad \forall g \in G'$$

The right-guarding LP can be formulated symmetrically. Let z^*, z^*_L, z^*_R be the optima for the original, left and the right guarding problem, respectively.

The following important claim was shown by Kolen and Tamir [12] from the property that constraint matrix for the left-guarding problem is totally balanced (see Elbassioni et al. [6]). In contrast to their approach, we give a simple procedure that, in $O(nm)$ time, returns an optimal set of guards for the left-guarding problem.

Lemma 2 *Let G_L and G_R be the optimum sets of guards for the left and right guarding problem, respectively. Then $|G_L| = z^*_L$ and $|G_R| = z^*_R$.*

Our combinatorial proof of Lemma 2. For simplicity of notation, we will just assume for this subsection that $G_0 = \emptyset$, and hence $G' = G$. We first give a combinatorial algorithm for the problem of guarding a set of points N from the left.

Algorithm 1 LEFT-GUARDING(T, N, G)

1. $A(p) \leftarrow \emptyset$, $\forall p \in N$
2. **for** $p \in N$ processed from *left to right* **do**
3. **while** the number of guards in $A = \cup_{p\in N} A(p)$ that see p is less than $d_{p,L}$ **do**
4. $A(p) = A(p) \cup \{L(p)\}$
5. **return** A

In the algorithm, with $A(p)$ we denote the set of guards activated to satisfy the demand of the point p, and with $L(p)$ we denote the leftmost guard in the set $S_L(p)\backslash A$.

For the purpose of the analysis, we distribute the dual updates as follows:

Algorithm 2 DUAL-UPDATES$(\{A(p)\}, T)$

1. $y_p = 0$, $\forall p \in N$, $z_g = 0$, $\forall g \in G$
2. $x_g = \begin{cases} 1, & g \in A = \cup_{p\in N} A(p) \\ 0, & \text{otherwise} \end{cases}$
3. **for** $p \in N$ processed from *right to left* **do**
4. **if** $A(p)$ has a non-marked guard **then**
5. mark all guards in $S_L(p)$
6. $y_p = 1$
7. $z_g = \sum_{p|g\in S_L(p)} y_p - 1$, $\forall g \in A$

We assume that all the guards are initially unmarked. We will first argue that the primal and the dual solutions constructed above are both feasible.

Primal feasibility. Follows from line (3) of the *Algorithm 1.*

Dual feasibility. It is enough to show that $\sum_{p|g\in S_L(p)} y_p \ge 1$, $\forall g \in A$, and $\sum_{p|g\in S_L(p)} y_p \le 1$, $\forall g \in G \setminus A$. For the first claim, suppose that $g \in A(p)$ and $y_p = 0$. Then there should exist some $p' > p$ such that p' marked g and hence $y_{p'} = 1$ and $g \in S_L(p')$. For the second claim, consider some guard $\bar{g} \notin A$ and suppose that there are two points $p < p'$ such that $\bar{g} \in S_L(p) \cap S_L(p')$ and $y_p = y_{p'} = 1$. Since $y_p = 1$ there exists a guard $g < \bar{g}$ such that $g \in A(p)$ and g was unmarked when *Algorithm 2* was processing p. By the order claim it follows that g must also see p' and, therefore, cannot be unmarked.

We have found an integer feasible solution of the primal and an integer feasible solution of the dual problem. All that is left to prove that these solutions are optimal is to show that the complementary-slackness conditions hold.

Primal complementary-slackness. We need to show that

$$y_p = 1 \;\Rightarrow\; \sum_{g\in S_L(p)} x_g = d_{p,L}$$
$$z_g > 0 \;\Rightarrow\; x_g = 1.$$

If $z_g > 0$ then g is in A and hence $x_g = 1$. Suppose that $y_p = 1$. Then we want to claim that $\sum_{g\in S_L(p)} x_g = d_p$. Since $y_p = 1$, there exists a guard $g \in A(p)$ that was unmarked when p was being processed in the *Algorithm 2*. Suppose that $\sum_{g\in S_L(p)} x_g > d_p$. Then there is a $g' > g$ such that $g' \sim p$ and $g' \in A(p')$ for some $p' > p$. Suppose that guard g' is marked. But then the point that marked g' must also mark g by the order claim. On the other hand, if g' is not marked, then the point p' that is to the right of p (and, therefore, processed before point p), would have marked it since $g' \in A(p')$ and is not marked, together with g.

Dual complementary-slackness We need to show that $x_g = 1 \Rightarrow \sum_{p|g\in S_L(p)} y_p - z_g = 1$. This follows from step 7 of *Algorithm 2.* □

We conclude with the final theorem.

Theorem 3 *There is a 6.7-approximation for the 1.5D guarding problem with demands.*

Proof. Note first that $G_0 \cup G_L \cup G_R$ is a feasible solution, since each point p is seen by at least d_p guards from this set. Indeed, if $p \notin N'$ then p is already covered by G_0. If $p \in N'$, then $\sum_{g\in S'_L(p)} x^*_g + \sum_{g\in S'_R(p)} x^*_g + x^*_p \ge d_p - \sum_{g\in S(p)\cap G_0} x^*_g \ge d_p - |S(p) \cap G_0| = d'_p$, from which follows $d_{p,L} + d_{p,R} + |S(p) \cap G_0| \ge$

d_p, implying feasibility for point p, by the feasibility of G_L and G_R for the left and right subproblems, respectively.

Now we bound the approximation ratio. Note first by the definition of G_0 that $|G_0| \leq \frac{1}{\alpha}\sum_{g\in G_0} x_g^*$. We will show next that the restriction of $(1+\beta)x^*$ on G' is feasible for LP2, for some positive constant β. This will imply that $z_L^* \leq (1+\beta)\sum_{g\in G'} x_g^*$. By a similar argument, we can also show that $z_R^* \leq (1+\beta)\sum_{g\in G'} x_g^*$.

Namely, note that $\forall p \in N_L$ it holds that $\sum_{g\in S'_L(p)} x_g^* \geq \frac{1}{2}(d_{\min}-\alpha)$, and thus

$$\begin{aligned}\sum_{g\in S'_L(p)}(1+\beta)x_g^* &= \sum_{g\in S'_L(p)} x_g^* + \beta\cdot\sum_{g\in S'_L(p)} x_g^* \\ &\geq \sum_{g\in S'_L(p)} x_g^* + \beta\cdot\frac{1}{2}(d_{\min}-\alpha) \\ &\geq \sum_{g\in S'_L(p)} x_g^* + 1,\end{aligned}$$

where the last inequality follows for $\beta \geq 2/(d_{\min}-\alpha)$.

Moreover, $\forall p \in N_R$ by the fact that

$$d_{p,L} = \lfloor \sum_{g\in S'_L(p)} x_g^* + x_p^* \rfloor \leq \sum_{g\in S'_L(p)} x_g^* + \alpha,$$

it is enough to show the following

$$\sum_{g\in S'_L(p)}(1+\beta)x_g^* \geq \sum_{g\in S'_L(p)} x_g^* + \alpha \qquad (2)$$

Using $\sum_{g\in S'_L(p)} x_g^* \geq 1-\alpha$ (since otherwise, $d_{p,L}=0$), inequality (2) is satisfied if $\beta\cdot(1-\alpha)\geq\alpha$.

Finally note that for all $g\in G'$ the inequality $(1+\beta)x_g^* \leq 1$ will be satisfied if $\beta \leq \frac{1}{\alpha}-1$.

Hence, the cost of the returned solution is

$$\begin{aligned}|G_0|+|G_L|+|G_R| &= |G_0| + z_L^* + z_R^* \\ &\leq \frac{1}{\alpha}\sum_{g\in G_0} x_g^* + 2\cdot(1+\beta)\sum_{g\in G'} x_g^* \\ &\leq \max\{\frac{1}{\alpha}, 2\cdot(1+\beta)\}z^* \\ &\leq \max\{\frac{1}{\alpha}, 2\cdot(1+\beta)\}OPT\end{aligned}$$

where OPT denotes the optimal integer solution to the original problem.

The above constraints on β imply that the approximation factor is bounded by

$$\gamma = \min_{\alpha\in(0,\alpha')}\max\{\frac{1}{\alpha}, \frac{4}{d_{min}-\alpha}+2, \frac{2\alpha}{1-\alpha}+2\} \qquad (3)$$

where $\alpha' = \min\{\frac{1}{2}, \frac{3+d_{\min}-\sqrt{(3+d_{\min})^2-4d_{\min}}}{2}\}$. Note that for $d_{\min}\geq 3$, $\alpha' = \frac{1}{2}$.

One can easily verify that for $d_{\min}=1$, the maximum value in (3) will be for $\alpha = 0.149$ that balances the terms $\frac{1}{\alpha}$ and $\frac{4}{1-\alpha}+2$. This leads to $\gamma = 6.7$, which concludes the proof of the theorem. □

Remark. With a more careful analysis of (3), one can express the approximation factor in terms of $d_{\min}$, for $d_{\min} < 5$. Moreover, for $d_{\min}\geq 5$ and $\alpha = 1/3$, the approximation factor will reduce to $\gamma = 3$.

References

[1] B. Ben-Moshe, M. J. Katz, and J. S. B. Mitchell. A constant-factor approximation algorithm for optimal 1.5d terrain guarding. volume 36, pages 1631–1647. SICOMP, 2007.

[2] C. Chekuri, K. L. Clarkson, and S. Har-Peled. On the set multi-cover problem in geometric settings. In *SCG*, pages 341–350, 2009.

[3] V. E.-C. Chen, D. and J. Urrutia. Optimal guarding of polygons and monotone chains. In *CCCG*, pages 133–138, 1995.

[4] K. L. Clarkson and K. R. Varadarajan. Improved approximation algorithms for geometric set cover. In *SCG*, pages 135–141, 2005.

[5] E. D. Demaine and J. O'Rourke. Open problems from cccg 2005. In *CCCG*, 2005.

[6] K. M. Elbassioni, E. Krohn, D. Matijevic, J. Mestre, and D. Severdija. Improved approximations for guarding 1.5-dimensional terrains. *Algorithmica*, to appear:x–x, September 2009.

[7] M. Gibson, G. Kanade, E. Krohn, and K. R. Varadarajan. An approximation scheme for terrain guarding. In *APPROX-RANDOM*, pages 140–148. Springer, 2009.

[8] J. King. Approximation algorithms for guarding 1.5-dimensional terrains (master's thesis). *McGill University*, 2005.

[9] J. King. A 4-approximation algorithm for guarding 1.5-dimensional terrains. In *LATIN*, pages 629–640, 2006.

[10] J. King. VC-dimension of visibility on terrains. In *CCCG*, 2008.

[11] J. King and E. Krohn. The complexity of guarding terrains. In *SODA*, pages 1580–1593. SIAM, 2009.

[12] A. Kolen and A. Tamir. *Covering Problems*. P.B. Mirchandani, R.L. Francis, eds., Discrete Location Theory, Wiley, 1990.

[13] N. H. Mustafa and S. Ray. PTAS for geometric hitting set problems via local search. In *SCG*, pages 17–22, New York, NY, USA, 2009. ACM.