Locating Lines and Hyperplanes

Fermat – Torricelli – Weberov problem

$$I = \{1, \dots, m\}$$
$$\Lambda = \{T_i(x_i, y_i) \in \mathbb{R}^2 : i \in I\}$$
$$w = \{w_i > 0 : i \in I\}, \quad W = \sum_{i=1}^{m} w_i$$

Median problem: odrediti $T_0(x_0, y_0) \in \mathbb{R}^2$ koji je rjesenje problema

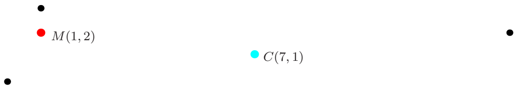
$$\sum_{i=1}^{m} w_i d(T_i, T) \longrightarrow \min_{T \in \mathbb{R}^2}$$

Problem centra: odrediti $T_0(x_0, y_0) \in \mathbb{R}^2$ koji je rjesenje problema

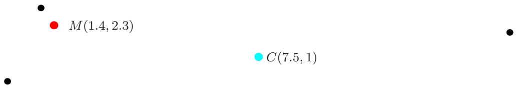
$$\max_{i=1,m} w_i d(T_i, T) \longrightarrow \min_{T \in \mathbb{R}^2}$$

Weiszfeld algoritam (1936) za: $d^{(2)}(T, T_i) = \sqrt{(x - x_i)^2 + (y - y_i)^2} =: d_i(x, y)$
Manhattan udaljenost za: $d^{\infty}(T, T_i) = |x - x_i| + |y - y_i|$

d_1: Medijan (crveno), Centar (zeleno)



d_2: Medijan (crveno), Centar (zeleno)



d_infty: Medijan (crveno), Centar (zeleno)

Γ – objekt (facility), primjerice: tocka, pravac (hiperravnina), segment, poligonalna linija, kruznica, graf

$$\sum_{i=1}^{m} w_i d(T_i, \Gamma) \longrightarrow \min_{\Gamma} \qquad \text{— \textbf{medijan problem}} \quad \text{ili problem sume}$$

$$\max_{i=1,m} w_i d(T_i, \Gamma) \longrightarrow \min_{\Gamma} \qquad \text{— \textbf{problem maksimuma}} \quad \text{ili problem centra}$$

Primjene:

- autoput, željeznička pruga (objekti – gradovi, tezine – broj stanovnika)
- trasa kanala za odvodnju i navodnjavanje, problem sirine skupa

Drugi pristupi:

- Numericka matematika – Computational Geometry (l_1 i l_∞ problem aproksimacije)
- Statistika (robust statistics, LAD regression, median problem, orthogonal regression)

p**-norma na** $\mathbb{R}^n$**:**

$$x \in \mathbb{R}^n, \quad \|x\|_p = \left(\sum_{i=1}^{n} w_i |x_i|^p \right)^{1/p}$$
$$\|x\|_\infty = \max_{i=1,n} w_i |x_i|$$

p**-udaljenost na** $\mathbb{R}^n$**:**

$$x, y \in \mathbb{R}^n, \quad d^p(x,y) = \|x-y\|_p = \left(\sum_{i=1}^{n} w_i |x_i - y_i|^p \right)^{1/p}$$
$$d^\infty(x,y) = \|x-y\|_\infty = \max_{i=1,n} w_i |x_i - y_i|$$

$p = 2$: Euklidska norma i udaljenost

$p = 1$: Manhattan (pravokutna) norma i udaljenost

$p = \infty$: Cebisevljeva max norma i udaljenost

Jedinicna kuglina ljuska na $\mathbb{R}^n$:

$$K^p(0,1) = \{x \in \mathbb{R}^n : d^p(0,x) = 1\} = \{x \in \mathbb{R}^n : \|x\|_p = 1\}$$

Ekstremne tocke jedinicne kugline ljuske na $\mathbb{R}^n$:

$$E(K^1) = \{e_i, -e_i : i = 1, \ldots, n\}$$
$$E(K^\infty) = \{(s_1, \ldots, s_n) \in \mathbb{R}^n : s_i \in \{-1, 1\}\}$$

Veza $d^1 \longleftrightarrow d^\infty$ na $\mathbb{R}^2$:

$$d^\infty(x,y) = d^1(Tx, Ty)$$

$$d^1(x,y) = d^\infty(T^{-1}x, T^{-1}y)$$

$T = \mathcal{A} \cdot \mathcal{U}$, $\quad \mathcal{A} = \frac{1}{\sqrt{2}}\mathcal{I}$, $\quad \mathcal{U}$ – rotacija zaa kut $(-\frac{\pi}{4})$

$$A(e) = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \qquad U(e) = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}, \qquad T(e) = \frac{1}{2} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$$

Hiperravnina zadana vektorom normale $\mathbf{n} = (s_1, \ldots, s_n)^T \in \mathbb{R}^n$, $\mathbf{n} \neq 0$ i parametrom $b \in \mathbb{R}$:

$$H_{\mathbf{n},b} := \{x \in \mathbb{R}^n : (x, \mathbf{n}) + b = 0\}, \qquad (x, \mathbf{n}) = x_1 s_1 + \cdots x_n s_n.$$

Hiperravnine u R^2 zadana vektorom normale $\mathbf{n} = (a, b)^T \in \mathbb{R}^2$, $a^2 + b^2 \neq 0$ i parametrom $c \in \mathbb{R}$:

$$p_{a,b,c} := \{(x, y) \in \mathbb{R}^2 : ax + by + c = 0\}, \text{ – pravac s koeficijentom smjera } \left(-\tfrac{a}{b}\right) \in \mathbb{R} \cup \{\infty\}.$$

Skup svih pravaca u ravnini moze se jednoznacno opisati s

$$ax + by + c = 0, \qquad a^2 + b^2 = 1, \tag{1}$$

i rastaviti na dva disjunktna podskupa:

$(b = 0)$ **Vertikalni pravci:** $p_{-1,0,c}$, tj. $x = c$

$(b \neq 0)$ **Ne-vertikalni pravci:** $p_{a',-1,b'}$, tj. $y = a'x + b'$

Hiperravninu H definira particiju skupa $\mathbb{R}^n = H \cup B_H^- \cup B_H^+$.

Horizontalna udaljenost na $\mathbb{R}^n$:

$$d_h(x,y) = \begin{cases} |y_1 - x_1|, & x_i = y_i,\ i = 2,\dots,n \\ \infty, & \text{inace} \end{cases}$$

Vertikalna udaljenost na $\mathbb{R}^n$:

$$d_v(x,y) = \begin{cases} |y_n - x_n|, & x_i = y_i,\ i = 1,\dots,n-1 \\ \infty, & \text{inace} \end{cases}$$

Primjer

Udaljenost d_v tocke $T_0(x_0, y_0)$ do ne-vertikalnog pravca $p_{a,-1,b}$ je

$$\begin{aligned} d_v(T_0, p) &= \min_{T \in p} d_v(T_0, T) \\ &= d_v\left(T_0, (x_0, ax_0 + b)\right) \\ &= |ax_0 + b - y_0| \end{aligned}$$

Veza $d_h \longleftrightarrow d_v$ na $\mathbb{R}^2$:

$$d_v(x,y) = d_h(Tx, Ty)$$
$$d_h(x,y) = d_v(Tx, y)$$

$$T - \text{rotacija za kut } (-\tfrac{\pi}{2}), \qquad T(e) = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

Lema

Let (w_i, y_i), $i \in I$, $I = \{1, \ldots, m\}$, $m \geq 2$, *be the data, whereby* $y_1 \leq y_2 \leq \ldots \leq y_m$ *are real numbers, and* $w_i > 0$ *are corresponding weights, and* $W = \sum_{i=1}^{m} w_i$. *Denote*

$$\nu_0 = \max \left\{ \nu \in I : \sum_{i=1}^{\nu} w_i \leq \frac{W}{2} \right\}.$$

Furthermore, let $F : \mathbb{R} \to \mathbb{R}$ *be a function defined by the formula*

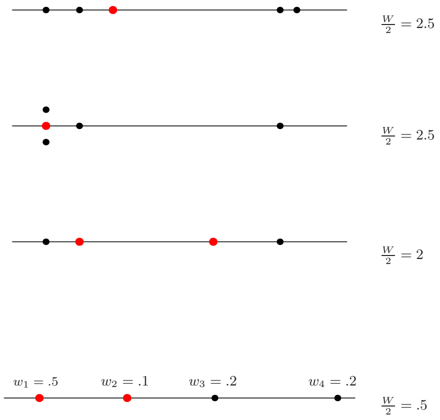
$$F(\alpha) = \sum_{i=1}^{m} w_i |y_i - \alpha|. \tag{2}$$

(i) *If* $\sum_{i=1}^{\nu_0} w_i < \frac{W}{2}$, *then the minimum of the function* F *is attained at the point* $\alpha^\star = y_{\nu_0+1}$.

(ii) *If* $\sum_{i=1}^{\nu_0} w_i = \frac{W}{2}$, *then the minimum of the function* F *is attained at every point* $\alpha^\star$ *from the segment* $[y_{\nu_0}, y_{\nu_0+1}]$.

Primjedba. Primijetite da kao posljedica ove leme neposredno slijedi **pseudo–halving property** podataka (w_i, y_i), $i \in I$, pa mozemo reci da je median ovih podataka svaki broj α^* za kojeg vrijedi

$$\sum_{y_i < \alpha^*} w_i \leq \frac{W}{2} \quad \text{and} \quad \sum_{y_i > \alpha^*} w_i \leq \frac{W}{2}. \tag{3}$$



$\frac{W}{2} = 2.5$
$\frac{W}{2} = 2.5$
$\frac{W}{2} = 2$
$w_1 = .5$
$w_2 = .1$
$w_3 = .2$
$w_4 = .2$
$\frac{W}{2} = .5$

Hiperplane location problem

Med 1 **(Weak incidence property):** postoji optimalna median hiperravnina koja prolazi kroz n afino nezavisnih tocaka iz Λ;

Med 2 **(Pseudo–halving property):** za svaku optimalnu median hiperravninu H vrijedi

$$\sum_{T_i \in B_H^-} w_i \leq \frac{W}{2} \quad \text{and} \quad \sum_{T_i \in B_H^+} w_i \leq \frac{W}{2};$$

Cen 1 **(Weak blockedness property):** postoji optimalna max (centar) hiperravnina H koja ima svojstvo da je barem $n+1$ tocaka iz Λ tezinski maksimalno udaljeno od H;

Cen 2 **(Paralel facets property):** ako je $w_1 = \cdots = w_m$, onda postoji optimalna max (centar) hiperravnina koja je paralelna stranici konveksne ljuske tocaka skup Λ;

Cen 1' **(Modified weak blockedness property):** postoji optimalna max (centar) hiperravnina H koja ima svojstvo da je $n+1$ afino nezavisnih tocaka iz Λ tezinski maksimalno udaljeno od H;

Line location with vertical distances

Teorem

Let $I = \{1, \ldots, m\}$, $m \geq 2$ *be a set of indices,* $\Lambda = \{T_i(x_i, y_i) \in \mathbb{R}^2 : i \in I\}$ *a set of points in the plane, such that* $\min\limits_{i \in I} x_i < \max\limits_{i \in I} x_i$, *and* $w_i > 0$ *corresponding data weights. Then*

(i) *there exists the best weighted LAD-line* $f(x) = a^* x + b^*$ *which passes through at least two different points from* Λ*;*

(ii) *the best weighted LAD-line* $f(x) = a^* x + b^*$ *is pseudo–halving*

Dualna interpretacija

X – skup svih točcaka i pravaca u ravnini M;

$\mathcal{A} : X \to X$ – preslikavanje, zadano na sljedeći način:

$\mathcal{A}(p_{a,-1,b}) = (a, b)$

$\mathcal{A}(T_0(x_0, y_0)) = q_{-x_0,-1,y_0}$

Vrijedi:

$$d_v(T_0, p) = d_v\left(\mathcal{A}(T_0), \mathcal{A}(p)\right)$$

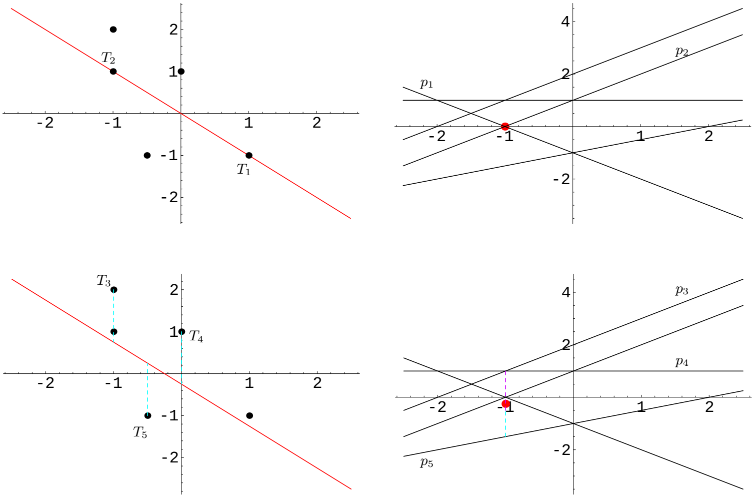
$$\begin{aligned} d_v(T_0, p) &= |ax_0 + b - y_0| = |-ax_0 + y_0 - b| \\ &= d_v\left(q_{-x_0,-1,y_0}, (a, b)\right) \\ &= d_v\left(\mathcal{A}(T_0), \mathcal{A}(p)\right) \end{aligned}$$

Vrijedi:

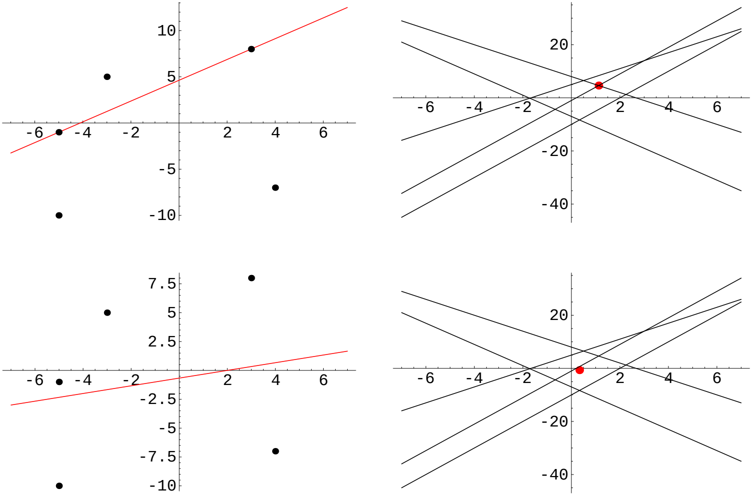
$$\sum_{i=1}^{m} d_v(T_i, p) = \sum_{i=1}^{m} d_v\left(\mathcal{A}(T_i), \mathcal{A}(p)\right) \quad \cdots \quad l_1$$

$$\max_{i=1,m} d_v(T_i, p) = \max_{i=1,m} d_v\left(\mathcal{A}(T_i), \mathcal{A}(p)\right) \quad \cdots \quad l_\infty$$

$\{\{1,-1\},\{-1,1\},\{-1,2\},\{0,1\},\{-0.5,-1\}\}$
Median (l_1) $\{3.5,\{a\to-1,b\to0\}\}$
Centar (l_infty) $\{1.25,\{a\to-1.,b\to-0.25\}\}$



$\{\{-5,-1\},\{4,-7\},\{-3,5\},\{3,8\},\{-5,-10\}\}$
$\{28.875,\{a \to 1.125, b \to 4.625\}\}$
$\{7.66667,\{a \to 0.333333, b \to -0.666667\}\}$



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